On the Role of the Growth Optimal Portfolio in Finance

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Introduction

- **mean-variance portfolio selection**

  Markowitz (1952, 1959)

  Markowitz efficient frontier

- **growth optimal portfolio** (GOP)

  Kelly (1956), Latané (1959)

  Breiman (1961)

- **capital asset pricing model** (CAPM)

  Sharpe (1964), Lintner (1965), Mossin (1966), Merton (1973)

- **Sharpe ratio**

  Sharpe (1964)

- **arbitrage pricing theory**

  Black & Scholes (1973), Merton (1973), Ross (1976), Harrison & Kreps (1979), Harrison & Pliska (1981), ...

- **numeraire portfolio**

  Long (1990)

- **fair benchmark pricing**

  Platen (2002), ...

  benchmark approach

Continuous Benchmark Model

- **traded uncertainty**

  independent standard Wiener processes $W^1, W^2, \dots, W^d$
  $(\Omega, \mathcal{A}_T, \underline{\mathcal{A}}, P)$

- **primary security accounts**

$$dS_t^{(j)} = S_t^{(j)} \left(a_t^j \, dt + \sum_{k=1}^{d} b_t^{j,k} \, dW_t^k \right)$$

$t \in [0, T]$, $j \in \{0, 1, \dots, d\}$

- **savings account**

$$S_t^{(0)} = \exp\left\{\int_0^t r_s\,ds\right\}$$

- **market price of risk**

$$\theta_t = (\theta_t^1, \theta_t^2, \ldots, \theta_t^d)^\top$$

$$b_t\,\theta_t = [a_t - r_t\,\mathbf{1}]$$

Assumption 1 $b_t = [b_t^{j,k}]_{j,k=1}^d$ *is* invertible.

$\Longrightarrow$

market price of risk

$$\theta_t = b_t^{-1} [a_t - r_t \, \mathbf{1}]$$

- jth **primary security account**

$$dS_t^{(j)} = S_t^{(j)} \left(r_t \, dt + \sum_{k=1}^{d} b_t^{j,k} \, \left[\theta_t^k \, dt + dW_t^k \right] \right)$$

Portfolios

- **strategy** $\delta = \{\delta_t = (\delta_t^0, \ldots, \delta_t^d)^\top,\, t \in [0, T]\}$

  predictable, S-integrable

- **portfolio**

$$S_t^{(\delta)} = \sum_{j=0}^{d} \delta_t^j \, S_t^{(j)}$$

- **self-financing** $\Longrightarrow$

$$dS_t^{(\delta)} = \sum_{j=0}^{d} \delta_t^j \, dS_t^{(j)}$$

- **fraction**

$$\pi^j_{\delta,t} = \delta^j_t \, \frac{S^{(j)}_t}{S^{(\delta)}_t}$$

$j \in \{0, 1, \ldots, d\}$

$$\sum_{j=0}^{d} \pi^j_{\delta,t} = 1$$

- **strictly positive portfolio**

$$dS^{(\delta)}_t = S^{(\delta)}_t \left(r_t \, dt + \sum_{k=1}^{d} \sum_{j=1}^{d} \pi^j_{\delta,t} \, b^{j,k}_t \left(\theta^k_t \, dt + dW^k_t \right) \right)$$

- **logarithm of portfolio**

$$d\ln(S_t^{(\delta)}) = g_t^\delta\, dt + \sum_{k=1}^{d}\sum_{j=1}^{d} \pi_{\delta,t}^j\, b_t^{j,k}\, dW_t^k$$

- **growth rate**

$$g_t^\delta = r_t + \sum_{k=1}^{d}\sum_{j=1}^{d} \pi_{\delta,t}^j\, b_t^{j,k} \left(\theta_t^k - \frac{1}{2}\sum_{j=1}^{d} \pi_{\delta,t}^j\, b_t^{j,k}\right)$$

- **growth optimal portfolio**

Definition 2 ***GOP*** maximizes growth rate

$$g_t^{\delta_*} \geq g_t^\delta.$$

(maximizes expected logarithmic utility) first order conditions $\Longrightarrow$

$$0 = \sum_{k=1}^{d} b_t^{j,k} \left(\theta_t^k - \sum_{\ell=1}^{d} \pi_{\delta,t}^{\ell} \, b_t^{\ell,k} \right)$$

$\Longrightarrow$

$$\pi_{\delta,t}^{\top} \, b_t = \theta_t^{\top}, \qquad \pi_{\delta_*,t} = \left(b_t^{-1} \right)^{\top} \theta_t$$

- **GOP SDE**

$$dS_t^{(\delta_*)} = S_t^{(\delta_*)} \left(\left[r_t + \sum_{k=1}^{d} (\theta_t^k)^2 \right] dt + \sum_{k=1}^{d} \theta_t^k \, dW_t^k \right)$$

$\Longrightarrow$ **continuous benchmark model**

existence of equivalent risk neutral measure ***not*** required

Portfolio Selection

- **discounted portfolio**

$$\bar{S}_t^{(\delta)} = \frac{S_t^{(\delta)}}{S_t^{(0)}}$$

$$d\bar{S}_t^{(\delta)} = \psi_{\delta,t}^\top \{\theta_t \, dt + dW_t\}$$

with

$$\psi_{\delta,t}^\top = (\psi_{\delta,t}^1, \ldots, \psi_{\delta,t}^d) = \bar{S}_t^{(\delta)} \, \pi_{\delta,t}^\top \, b_t$$

- **discounted drift**

$$\alpha_t^{\delta} = \psi_{\delta,t}^{\top}\,\theta_t$$

- **aggregate diffusion coefficient**

$$\gamma_t^{\delta} = \sqrt{\psi_{\delta,t}^{\top}\,\psi_{\delta,t}}$$

Definition 3 *portfolio* optimal *if*

$$\gamma_t^{\tilde{\delta}} = \gamma_t^{\delta}$$
$$\alpha_t^{\tilde{\delta}} \geq \alpha_t^{\delta}.$$

"more rather than less"

Assumption 4

$$|\theta_t| = \sqrt{\theta_t^\top \, \theta_t} > 0,$$
$$\pi^0_{\delta_*,t} \neq 1$$

- **Portfolio Selection Theorem**

$S^{(\delta)}$ optimal $\Longleftrightarrow$

$$d\bar{S}^{(\delta)}_t = \bar{S}^{(\delta)}_t \, \frac{1 - \pi^0_{\delta,t}}{1 - \pi^0_{\delta_*,t}} \; \theta_t^\top \, (\theta_t \, dt + dW_t)$$

with

$$\pi_{\delta,t} = \frac{1 - \pi^0_{\delta,t}}{1 - \pi^0_{\delta_*,t}} \, \pi_{\delta_*,t}$$

GOP interpreted as a *mutual fund*

Risk Aversion Coefficient

$$J_\delta(t, \bar{S}_t^{(\delta)}) = \frac{1 - \pi^0_{\delta_*,t}}{1 - \pi^0_{\delta,t}}$$

Arrow-Pratt absolute risk aversion coefficient

$\Longrightarrow$ **optimal portfolio fractions**

$$\pi_{\delta,t} = \frac{\pi_{\delta_*,t}}{J_\delta(t, \bar{S}_t^{(\delta)})}$$

for a GOP

$$J_{\delta_*}(t, \bar{S}_t^{(\delta_*)}) = 1$$

Utility Maximization

- **expected utility**

$$E\left(U(\bar{S}_T^{(\delta)})\,\Big|\,\bar{S}_t^{(\delta)}=x\right)$$

- **value function**

$$u(t,x)=\sup_{\pi_\delta\in\mathcal{V}_t} E\left(U(\bar{S}_T^{(\delta)})\,\Big|\,\bar{S}_t^{(\delta)}=x\right)$$

Krylov (1980), Cox & Huang (1989),
Fleming & Soner (1992), Korn (1997),
Karatzas & Shreve (1998)

- **Hamilton-Jacobi-Bellman equation**

$$\frac{\partial u(t,x)}{\partial t}+\sup_{\pi_\delta \in \mathcal{V}_t}\left(\pi_{\delta,t}^\top\, b_t\, \theta_t\, x\, \frac{\partial u(t,x)}{\partial x}+\frac{1}{2}\,\pi_{\delta,t}^\top\, b_t\, b_t^\top\, \pi_{\delta,t}\, x^2\, \frac{\partial^2 u(t,x)}{\partial x^2}\right)=0$$

terminal condition

$$u(T,x)=U(x)$$

- **first order conditions**

$$\frac{\partial}{\partial \pi_{\delta,t}^j}\left[\sum_{k=1}^d \sum_{\ell=1}^d \pi_{\delta,t}^\ell\, b_t^{\ell,k}\, \theta_t^k\, x\, \frac{\partial u(t,x)}{\partial x}+\frac{1}{2}\sum_{k=1}^d\left(\sum_{\ell=1}^d \pi_{\delta,t}^\ell\, b_t^{\ell,k}\right)^2 x^2\, \frac{\partial^2 u(t,x)}{\partial x^2}\right]=0$$

$\Longrightarrow$

$$b_t\, x \left[\theta_t\, \frac{\partial u(t,x)}{\partial x} + b_t^\top\, \pi_{\delta,t}\, x\, \frac{\partial^2 u(t,x)}{\partial x^2}\right] = (0,\dots,0)^\top .$$

substitute optimal portfolio $\quad\Longrightarrow$

$$b_t\, x\, \theta_t \left[\frac{\partial u(t,x)}{\partial x} + \frac{x}{J_\delta(t,x)}\, \frac{\partial^2 u(t,x)}{\partial x^2}\right] = (0,\dots,0)^\top .$$

$\Longrightarrow$ **risk aversion coefficient**

$$J_\delta(t,x) = -x\, \frac{\frac{\partial^2 u(t,x)}{\partial x^2}}{\frac{\partial u(t,x)}{\partial x}} .$$

Arrow-Pratt absolute risk aversion coefficient

- **classical Merton problem**

  Merton (1971)
  power utility $U(x) = \frac{1}{\gamma}\, x^{\gamma}$

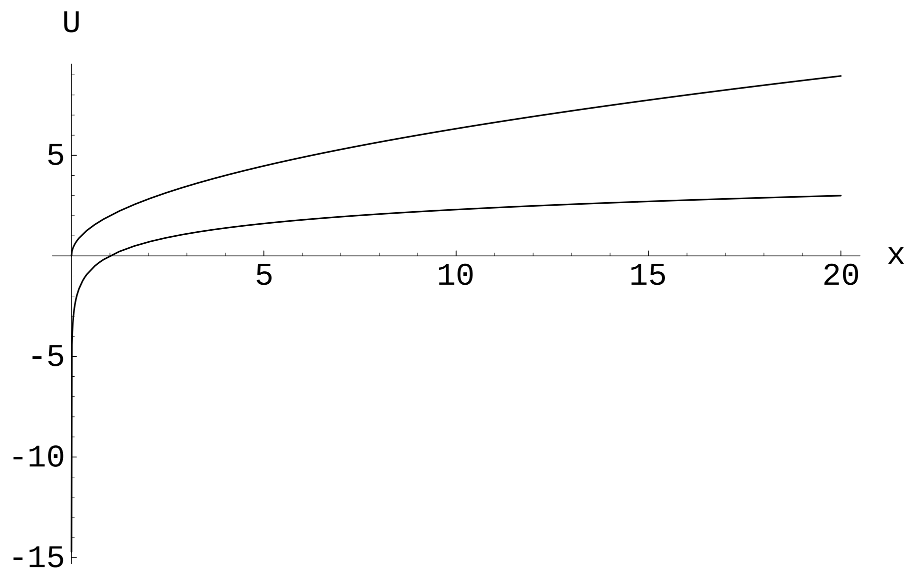

Figure 1: Examples for power and logarithmic utility.

- **value function**

$$u(t,\bar{S}_t^{(\delta)}) = \exp\left\{-\int_t^T \frac{|\theta_s|^2\,\gamma}{2\,(1-\gamma)} ds\right\} \left(\bar{S}_t^{(\delta)}\right)^{\gamma}$$

optimal fractions

$$\pi_{\delta,t} = \left(b_t^{-1}\right)^{\top} \theta_t \, \frac{1}{1-\gamma}$$

risk aversion coefficient

$$J_\delta(t,\bar{S}_t^{(\delta)}) = 1-\gamma$$

$\gamma \to 0$ GOP

$$u(t,\bar{S}_t^{(\delta)}) = \ln\left(\bar{S}_t^{(\delta)}\right) + \int_t^T \frac{|\theta_s|^2}{2}\, ds$$

- **market portfolio** $S^{(\delta_+)}$ investable wealth

Assumption 5 *Each investor forms optimal portfolio.*

$\Longrightarrow$

$$\begin{aligned} d\bar{S}_t^{(\delta_+)} &= d\left(\sum_{\ell=1}^{n} \bar{S}_t^{(\delta_\ell)}\right) \\ &= \sum_{\ell=1}^{n} \bar{S}_t^{(\delta_\ell)} \frac{1-\pi^0_{\delta_\ell,t}}{1-\pi^0_{\delta_*,t}}\ \theta_t^\top (\theta_t\,dt + dW_t) \\ &= \sum_{\ell=1}^{n} \left(\bar{S}_t^{(\delta_\ell)} - \delta^0_\ell(t)\right)\ \frac{\theta_t^\top}{1-\pi^0_{\delta_*,t}} (\theta_t\,dt + dW_t) \\ &= \bar{S}_t^{(\delta_+)} \frac{1-\pi^0_{\delta_+,t}}{1-\pi^0_{\delta_*,t}}\ \theta_t^\top (\theta_t\,dt + dW_t) \end{aligned}$$

$\Longrightarrow$ **market portfolio is an optimal portfolio**

risk aversion coefficient of market portfolio

$$J_{\delta_+}(t, \bar{S}_t^{(\delta_+)}) = \frac{1 - \pi^0_{\delta_*,t}}{1 - \pi^0_{\delta_+,t}}$$

- **is market portfolio a GOP ?**

- **market price of risk vector** $\theta_t = b_t^\top \, \pi_{\delta_*,t}$

- **risk premium vector** $p(t) = b_t \, b_t^\top \, \pi_{\delta_*,t}$

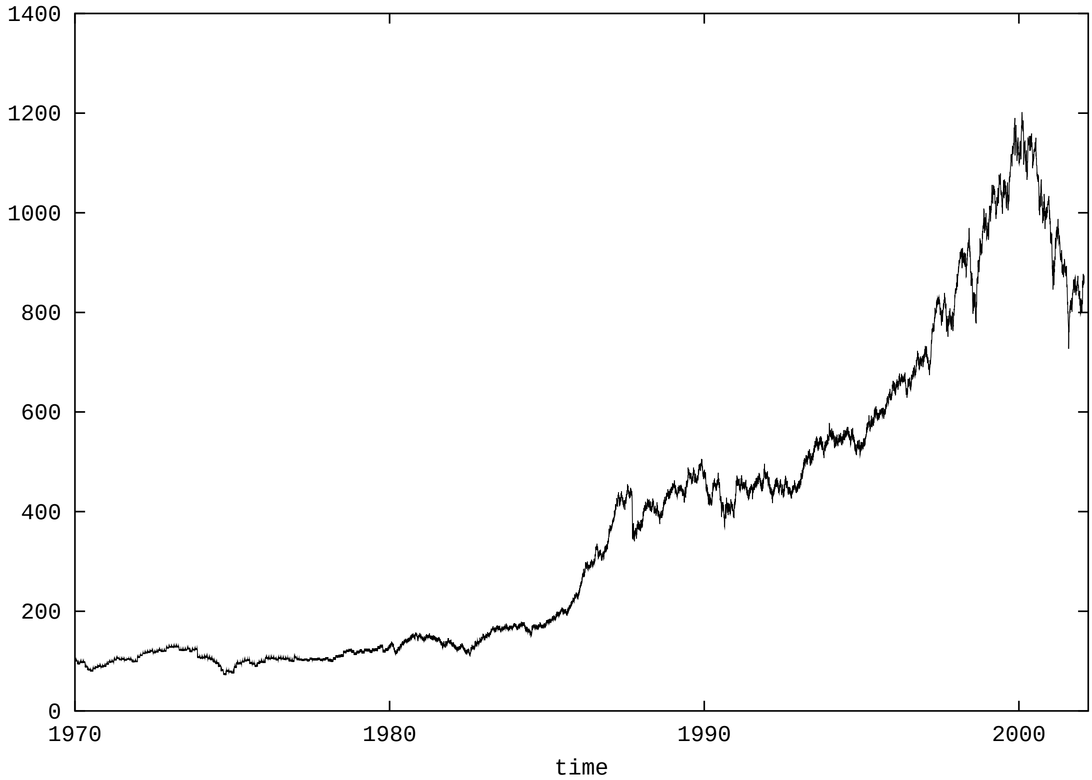

Figure 2: Discounted MSCI world index.

Risk Measurement

- **GOP volatilities** $\Longrightarrow$ **general market risk**

$$|\theta_t| = \sqrt{\sum_{k=1}^{d} (\theta_t^k)^2}$$

- **benchmarked security volatilities**

$$\hat{S}_t^{(j)} = \frac{S_t^{(j)}}{S_t^{(\delta_*)}}$$

$$d\hat{S}_t^{(j)} = -\hat{S}_t^{(j)} \sum_{k=1}^{d} \sigma_t^{j,k}\, dW_t^k$$

$$\sigma_t^{j,k} = \theta_t^k - b_t^{j,k}$$

$\Longrightarrow$ **specific market risk**

Capital Asset Pricing Model

- **risk premium** for $S^{(\delta)}$

$$p_\delta(t) = \sum_{k=1}^{d} \sum_{j=1}^{d} \pi^j_{\delta,t}\, b^{j,k}_t\, \theta^k_t$$

- **systematic risk parameter**

$$\beta_\delta(t) = \frac{\frac{d}{dt}\langle \ln(S^{(\delta)}), \ln(S^{(\delta_+)})\rangle_t}{\frac{d}{dt}\langle \ln(S^{(\delta_+)}), \ln(S^{(\delta_+)})\rangle_t}$$

if $S^{(\delta_+)}$ optimal portfolio
$\Longrightarrow$

$$\beta_\delta(t) = \frac{p_\delta(t)}{p_{\delta_+}(t)}$$

Efficient Frontier

- **optimal portfolio** $S^{(\delta)}$

  volatility

$$|b_\delta(t)| = \left| \frac{1 - \pi^0_{\delta,t}}{1 - \pi^0_{\delta_*,t}} \right| |\theta_t| = \frac{|\theta_t|}{J_\delta(t, \bar{S}^{(\delta)}_t)}$$

  risk premium

$$p_\delta(t) = \frac{1 - \pi^0_{\delta,t}}{1 - \pi^0_{\delta_*,t}} \, |\theta_t|^2 = |b_\delta(t)| \, |\theta_t|$$

- **appreciation rate**

$$a_\delta(t) = r_t + p_\delta(t) = r_t + \sqrt{|b_\delta(t)|^2} \, |\theta_t|$$

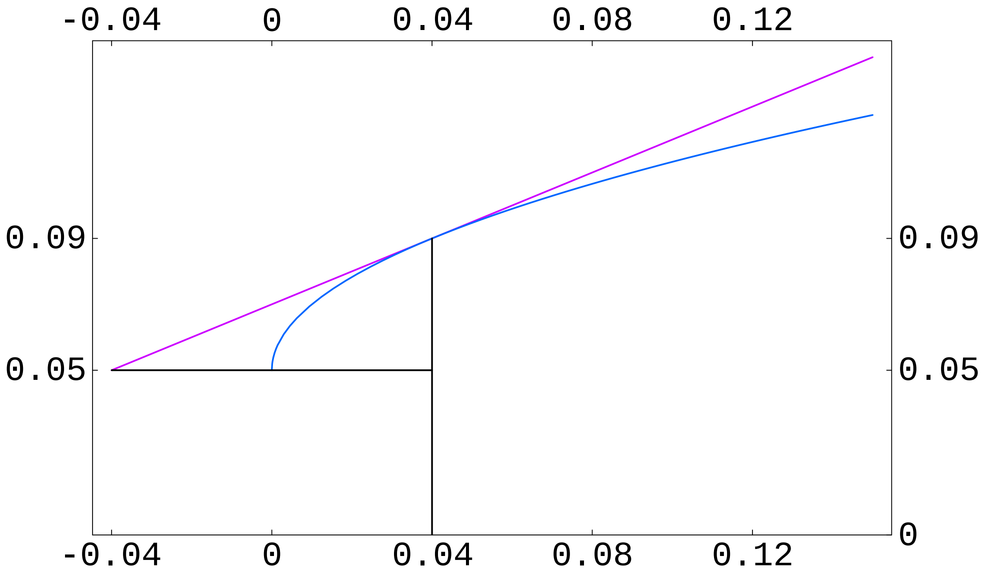

Figure 3: Efficient frontier.

- **Markowitz efficient frontier**

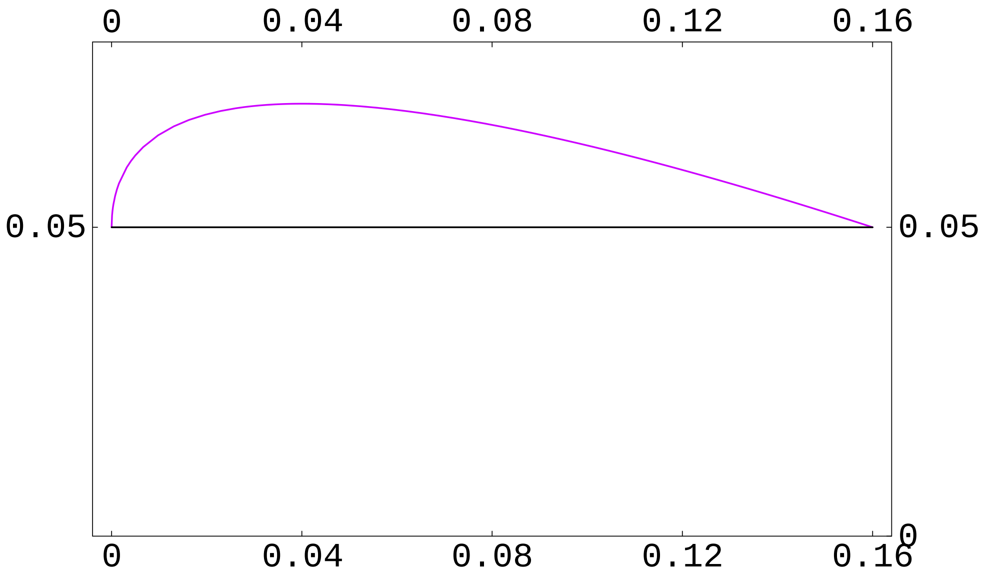

Figure 4: Efficient growth rates.

- **efficient growth rate**

$$g_\delta(t) = r_t + \sqrt{|b_\delta(t)|^2}\,|\theta_t| - \frac{1}{2}\,|b_\delta(t)|^2$$

Sharpe Ratio

risk premium $p_\delta(t)$

aggregate volatility $|b_\delta(t)|$

- **Sharpe ratio**

$$s_\delta(t) = \frac{p_\delta(t)}{|b_\delta(t)|} = \frac{\alpha_t^\delta}{\gamma_t^\delta}$$

$\Longrightarrow$ for any risky portfolio $S^{(\delta)}$

$$s_\delta(t) \leq |\theta_t|$$

equality for optimal portfolios

Benchmarked Portfolios

$$\hat{S}_t^{(\delta)} = \frac{S_t^{(\delta)}}{S_t^{(\delta_*)}}$$

$$d\hat{S}_t^{(\delta)} = -\sum_{k=1}^{d} \sum_{j=0}^{d} \delta_t^{(j)} \, \hat{S}_t^{(j)} \, \sigma_t^{j,k} \, dW_t^k$$

$(\underline{\mathcal{A}}, P)$-local martingale

$\Longrightarrow$

Nonnegative benchmarked portfolios are $(\underline{\mathcal{A}}, P)$-**supermartingales**.

Arbitrage

Definition:

Nonnegative portfolio $S^{(\delta)}$ permits arbitrage if for $S_0^{(\delta)} = 0$

$$P(S_T^{(\delta)} > 0) > 0.$$

$\Longrightarrow$

Continuous benchmark models do not permit arbitrage.

Equivalent local martingale measure may **not** exist.

Fair Contingent Claim Pricing

value process **fair** if its benchmarked value forms
$(\underline{\mathcal{A}}, P)$-martingale

- **fair pricing formula**

$$U_{H_\tau}(t) = S_t^{(\delta_*)} \, E\left(\frac{H_\tau}{S_\tau^{(\delta_*)}} \,\middle|\, \mathcal{A}_t \right)$$

Risk Neutral Pricing

Radon-Nikodym derivative

$$\Lambda_t = \frac{S_0^{(\delta_*)}}{S_t^{(\delta_*)}} \frac{S_t^{(0)}}{S_0^{(0)}} = \frac{\hat{S}_t^{(0)}}{\hat{S}_0^{(0)}}$$

fair pricing formula

$$\begin{aligned} U_{H_\tau}(t) &= E\left(\left(\frac{S_t^{(\delta_*)}}{S_\tau^{(\delta_*)}} \frac{S_\tau^{(0)}}{S_t^{(0)}}\right) \frac{S_t^{(0)}}{S_\tau^{(0)}} H_\tau \,\middle|\, \mathcal{A}_t\right) \\ &= E\left(\frac{\Lambda_\tau}{\Lambda_t} \frac{S_t^{(0)}}{S_\tau^{(0)}} H_\tau \,\middle|\, \mathcal{A}_t\right) \end{aligned}$$

If Λ is an $(\underline{\mathcal{A}}, P)$-martingale:

$$U_{H_\tau}(t) = E_\theta \left(\frac{S_t^{(0)}}{S_\tau^{(0)}} \, H_\tau \,\middle|\, \mathcal{A}_t \right)$$

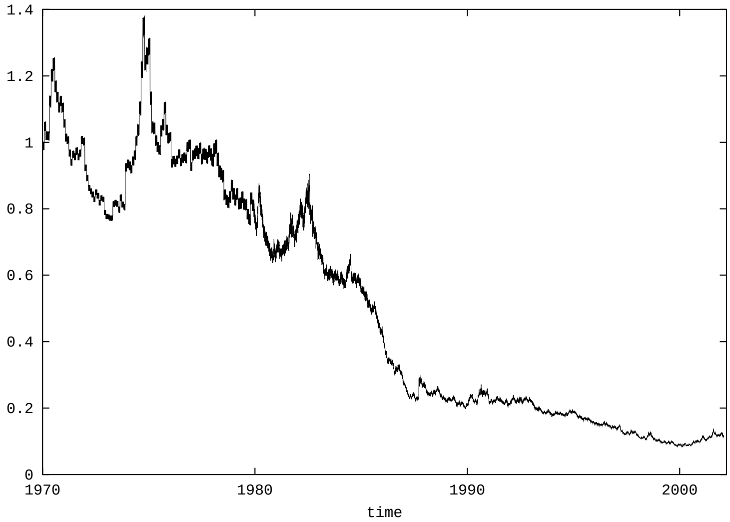

Figure 5: Radon-Nikodym derivative $\Lambda(t)$.

Outperformance by the GOP

- **GOP outperforms the growth rate**

- **expected return**

  nonnegative benchmarked portfolio

  $\hat{S}^{(\delta)}$ $\quad (\underline{\mathcal{A}}, P)$-supermartingale

  $\Longrightarrow$

$$E\left(\frac{\hat{S}^{(\delta)}_{t+h} - \hat{S}^{(\delta)}_t}{\hat{S}^{(\delta)}_t} \,\middle|\, \mathcal{A}_t\right) \leq 0$$

Definition:

positive portfolio $S^{(\delta)}$ **systematically outperforms** $S^{(\bar{\delta})}$

if for $\tau \in [0, T]$ and $\sigma \in [\tau, T]$ with

$$S_\tau^{(\delta)} = S_\tau^{(\bar{\delta})}$$

and

$$S_\sigma^{(\delta)} \geq S_\sigma^{(\bar{\delta})},$$

$$P\left(S_\sigma^{(\delta)} > S_\sigma^{(\bar{\delta})} \,\middle|\, \mathcal{A}_\tau\right) > 0.$$

$\Longrightarrow$

No nonnegative portfolio systematically outperforms the GOP.

- **long term growth rate**

$$\tilde{g}_\delta \stackrel{\text{a.s.}}{=} \limsup_{T \to \infty} \frac{1}{T} \, \ln \left(\frac{S_T^{(\delta)}}{S_0^{(\delta)}} \right)$$

- **GOP has greatest long term growth rate**

$$\tilde{g}_{\delta_*} \geq \tilde{g}_\delta$$

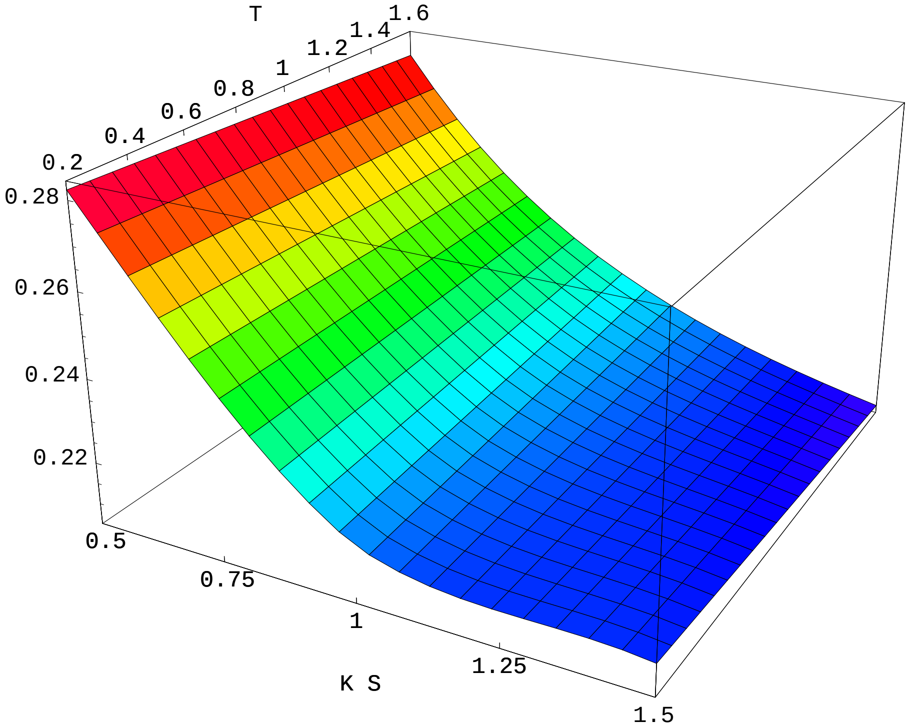

Figure 6: Average S&P500 implied volatility surface.

- **Discounted GOP**

$$d\bar{S}_t^{(\delta_*)} = \alpha_t \, dt + \sqrt{\alpha_t \, \bar{S}_t^{(\delta_*)}} \, dW_t$$

time transformed **squared Bessel process of dimension four**

- **Minimal Market Model**

  Platen (2001, 2002), Heath & Platen (2002, 2003, 2004)

$$\alpha_t = \alpha_0 \exp\left\{ \int_0^t \eta_s \, ds \right\} m_t$$

  η_t - net growth rate

  α_0 - initial discounted GOP drift

  m_t - market activity

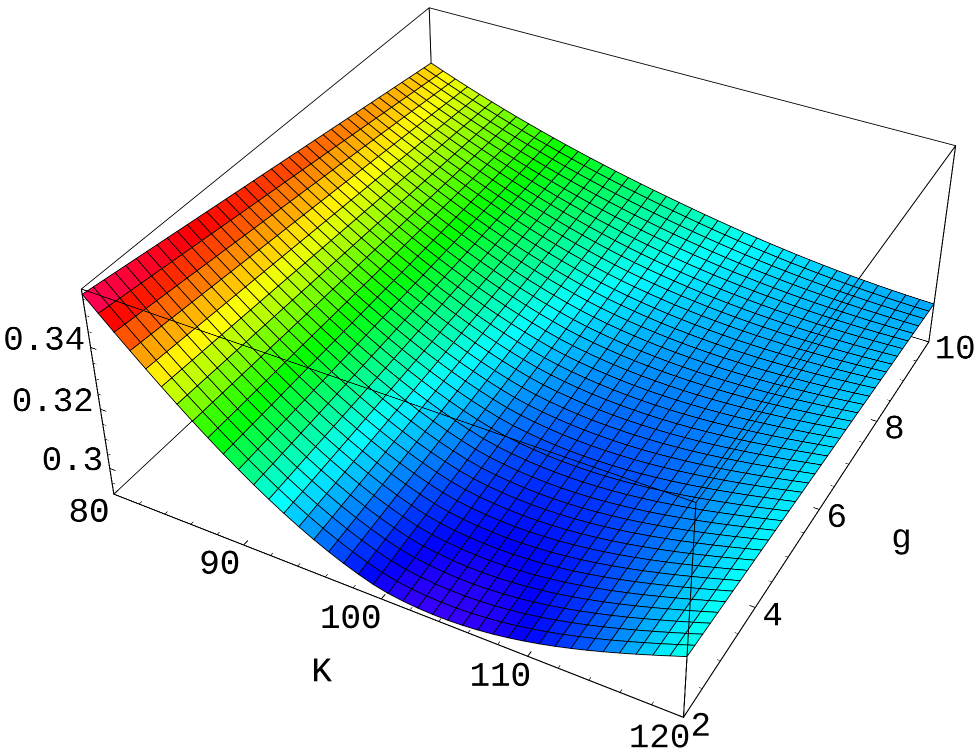

Figure 7: MMM implied volatility surface.

References

Black, F. & M. Scholes (1973). The pricing of options and corporate liabilities. *J. Political Economy* **81**, 637–654.

Breiman, L. (1961). Optimal gambling systems for favorable games. In *Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability*, Volume I, pp. 65–78.

Cox, J. C. & C. F. Huang (1989). Optimum consumption and portfolio policies when asset prices follow a diffusion process. *J. Economic Theory* **49**, 33–83.

Fleming, W. H. & H. M. Soner (1992). *Controlled Markov Processes and Viscosity Solutions.* Springer.

Harrison, J. M. & D. M. Kreps (1979). Martingale and arbitrage in multiperiod securities markets. *J. Economic Theory* **20**, 381–408.

Harrison, J. M. & S. R. Pliska (1981). Martingales and stochastic integrals in the theory of continuous trading. *Stochastic Process. Appl.* **11**(3), 215–260.

Heath, D. & E. Platen (2002). Perfect hedging of index derivatives under a minimal market model. *Int. J. Theor. Appl. Finance* **5**(7), 757–774.

Heath, D. & E. Platen (2003). Pricing of index options under a minimal market model with lognormal scaling. *Quant. Finance.* **3**(6), 442–450.

Heath, D. & E. Platen (2004). Understanding the implied volatility surface for options on a diversified index. Technical report, University of Technology, Sydney. QFRC Research Paper 128, to appear in *Asia-Pacific Financial Markets* **11**(1).

Karatzas, I. & S. E. Shreve (1998). *Methods of Mathematical Finance*, Volume 39 of *Appl. Math.* Springer.

Kelly, J. R. (1956). A new interpretation of information rate. *Bell Syst. Techn. J.* **35**, 917–926.

Korn, R. (1997). *Optimal Portfolios.* World Scientific.

Krylov, N. V. (1980). *Controlled Diffusion Processes*, Volume 14 of *Appl. Math.* Springer.

Latané, H. (1959). Criteria for choice among risky ventures. *J. Political Economy* **67**, 144–155.

Lintner, J. (1965). The valuation of risk assets and the selection of risky investments in stock portfolios and capital budgets. *Rev. Econom. Statist.* **47**, 13–37.

Long, J. B. (1990). The numeraire portfolio. *J. Financial Economics* **26**, 29–69.

Markowitz, H. (1952). Portfolio selection. *J. Finance* **VII**(1), 77–91.

Markowitz, H. (1959). *Portfolio Selection: Efficient Diversification of Investment.* Wiley, New York.

Merton, R. C. (1971). Optimum consumption and portfolio rules in a continuous-time model. *J. Economic Theory* **3**(4), 373–413.

Merton, R. C. (1973). Theory of rational option pricing. *Bell J. Econ. Management Sci.* **4**, 141–183.

Mossin, J. (1966). Equilibrium in a capital asset market. *Econometrica* **3**, 768–783.

Platen, E. (2001). A minimal financial market model. In *Trends in Mathematics*, pp. 293–301. Birkhäuser.

Platen, E. (2002). Arbitrage in continuous complete markets. *Adv. in Appl. Probab.* **34**(3), 540–558.

Ross, S. A. (1976). The arbitrage theory of capital asset pricing. *J. Economic Theory* **13**, 341–360.

Sharpe, W. F. (1964). Capital asset prices: A theory of market equilibrium under conditions of risk. *J. Finance* **19**, 425–442.