On the Role of the Growth Optimal Portfolio in Finance

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Abstract. The paper discusses various roles that the growth optimal portfolio (GOP) plays in finance. For the case of a continuous market we show how the GOP can be interpreted as a fundamental building block in financial market modeling, portfolio optimization, contingent claim pricing and risk measurement. On the basis of a portfolio selection theorem, optimal portfolios are derived. These allocate funds into the GOP and the savings account. A risk aversion coefficient is introduced, controlling the amount invested in the savings account, which allows to characterize portfolio strategies that maximize expected utilities. Natural conditions are formulated under which the GOP appears as the market portfolio. A derivation of the intertemporal capital asset pricing model is given without relying on Markovianity, equilibrium arguments or utility functions. Fair contingent claim pricing, with the GOP as numeraire portfolio, is shown to generalize risk neutral and actuarial pricing. Finally, the GOP is described in various ways as the best performing portfolio.

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1 Introduction

There exists an increasing literature on the *growth optimal portfolio* (GOP), which is the portfolio that maximizes expected utility from terminal wealth. It was originally discovered in Kelly (1956) and Latané (1959) and later studied and applied in Breiman (1961), Long (1990) and by many other authors. The current paper highlights some aspects of the central role that the GOP plays in finance.

The Nobel prize winning work by Markowitz (1952, 1959) on single-period *mean-variance portfolio selection* has provided the foundation for modern portfolio theory. In the context of dynamic investment planning in a continuous time setting the mean-variance approach has received little attention. The literature has focussed more on investors seeking to maximize expected utility, which is a departure from the mean-variance approach. In practice, few if any investors know their utility function. Furthermore, utility functions that are widely used for mathematical convenience may not necessarily come close to an adequate description of an investor's view on expected return towards risk.

Wide spread in the literature is the use of the Arrow-Pratt absolute risk aversion coefficient, see Luenberger (1997). In the current paper a *risk aversion coefficient* is introduced in a continuous time setting, which allows to achieve in a unified framework the main objectives of utility maximization and mean-variance portfolio selection. It will be shown that the *growth optimal portfolio* (GOP) plays a central role in this context and more generally in finance. Under natural assumptions the market portfolio of investable wealth is shown to form a proxy of the GOP. This permits the derivation of the *capital asset pricing model* (CAPM), as developed by Sharpe (1964), Lintner (1965), Mossin (1966) and Merton (1973a), without relying on Markovianity, expected utility maximization and equilibrium arguments. Similarly, the *Markowitz efficient frontier*, see Markowitz (1959), and the *Sharpe ratio*, see Sharpe (1964), can be directly derived in the given continuous time framework with the GOP as benchmark. The method employed has similarities to that proposed in Fleming & Stein (2004), which uses stochastic optimal control to derive certain characteristics for an economy.

The GOP plays a natural role as *numeraire portfolio*, see Long (1990), in the pricing of contingent claims with the real world probability measure as pricing measure. The *fair pricing concept* of the benchmark approach generalizes the risk neutral pricing method of the *arbitrage pricing theory*, see Black & Scholes (1973), Merton (1973b), Ross (1976), Harrison & Kreps (1979) and Harrison & Pliska (1981). The existence of an equivalent risk neutral pricing measure is *not* required. The prescribed benchmark approach, with the GOP as its central building block, widens the range of models that can be applied in finance.

In Section 2 the paper introduces a continuous benchmark model. Section 3 studies various aspects of portfolio optimization and introduces a risk aversion coefficient. It also discusses utility maximization, the intertemporal CAPM and the measurement of general and specific market risk. In Section 4 the pricing of contingent claims is performed without measure transformation. Finally, in Section 5 the GOP is shown in various ways to be the best performing portfolio.

2 Continuous Benchmark Model

2.1 Primary Security Accounts

For an illustration of the central role of the GOP in finance let us consider a continuous financial market model. Here $\mathcal{A}_t$ denotes the market information that is available at time $t \in [0, T]$, $T \in (0, \infty)$. More precisely, the market is modeled on a filtered probability space $(\Omega, \mathcal{A}_T, \underline{\mathcal{A}}, P)$ with filtration $\underline{\mathcal{A}} = (\mathcal{A}_t)_{t\in[0,T]}$, satisfying the usual conditions, see Karatzas & Shreve (1991). We consider d continuous sources of traded uncertainty, which are modeled by d independent standard Wiener processes W^1, W^2, ..., W^d.

The d sources of traded uncertainty are securitized by $d+1$ *primary security accounts*, which are typically cum-dividend share accounts of respective companies or bonds. For $j \in \{0, 1, \dots, d\}$ we denote by $S_t^{(j)}$ the value of the j*th primary security account* at time t, when expressed in units of the domestic currency. We emphasize that all dividends or interest payments are reinvested. To provide a model for continuous market dynamics we assume that the price $S_t^{(j)}$ at time t for the jth primary security account is the unique strong solution of the SDE

$$dS_t^{(j)} = S_t^{(j)} \left(a_t^j \, dt + \sum_{k=1}^{d} b_t^{j,k} \, dW_t^k \right) \tag{2.1}$$

for all $t \in [0, T]$ with $S_0^{(j)} > 0$ and $j \in \{0, 1, \dots, d\}$. Here we require the *appreciation rate* processes $a^j = \{a_t^j, \, t \in [0, T]\}$ and *volatility* processes $b^{j,k} = \{b_t^{j,k}, \, t \in [0, T]\}$ to satisfy usual measurability and integrability conditions for all $j \in \{0, 1, \dots, d\}$ and $k \in \{1, 2, \dots, d\}$. We assume the existence of a *savings account* $S^{(0)} = \{S_t^{(0)}, \, t \in [0, T]\}$, with

$$S_t^{(0)} = \exp\left\{ \int_0^t r_s \, ds \right\}, \tag{2.2}$$

$$b_t^{0,k} = 0 \tag{2.3}$$

for all $k \in \{1, 2, \dots, d\}$ and the *short rate*

$$a_t^0 = r_t \tag{2.4}$$

for $t \in [0, T]$. Note that the appreciation rates, volatilities and short rate can depend on sources of uncertainty other than those given by the Wiener processes $W^1, W^2, \dots, W^d$. This means that the market is not assumed to be complete.

By introducing the appreciation rate vector $a_t = (a_t^1, a_t^2, \ldots, a_t^d)^\top$, the unit vector $\mathbf{1} = (1, 1, \ldots, 1)^\top$ and the *market price for risk* vector

$$\theta_t = (\theta_t^1, \theta_t^2, \ldots, \theta_t^d)^\top$$

it is reasonable to suppose that the equation

$$b_t\, \theta_t = [a_t - r_t\, \mathbf{1}] \tag{2.5}$$

has a unique solution. This is guaranteed, for instance, by the following natural assumption.

Assumption 2.1 *The volatility matrix $b_t = [b_t^{j,k}]_{j,k=1}^d$ is* invertible *for each $t \in [0, T]$.*

We directly obtain by (2.5) the market price for risk vector in the form

$$\theta_t = b_t^{-1}\, [a_t - r_t\, \mathbf{1}] \tag{2.6}$$

for $t \in [0, T]$. Without loss of generality, this allows us to rewrite the SDE (2.1) for the jth primary security account $S_t^{(j)}$ in the form

$$dS_t^{(j)} = S_t^{(j)} \left(r_t\, dt + \sum_{k=1}^d b_t^{j,k} \; \left[\theta_t^k\, dt + dW_t^k\right] \right) \tag{2.7}$$

for $t \in [0, T],\ j \in \{0, 1, \ldots, d\}$. For simplicity, we have chosen a continuous market model. However, note that the majority of the results that will be presented can be generalized to jump diffusion markets.

2.2 Portfolios

A stochastic process $\delta = \{\delta_t = (\delta_t^0, \ldots, \delta_t^d)^\top,\ t \in [0, T]\}$ is called a *strategy* if δ is predictable and S-integrable with respect to the vector process $S = \{S_t = (S_t^{(0)}, S_t^{(1)}, \ldots, S_t^{(d)})^\top,\ t \in [0, T]\}$ of primary security accounts. Here δ_t^j is the number of units of the jth primary security account that is held at time $t \in [0, T]$ and $j \in \{0, 1, \ldots, d\}$. A negative δ_t^j means that one is short δ_t^j units of the jth primary security account. For a strategy δ we denote by $S_t^{(\delta)}$ the value of the corresponding *portfolio* or *wealth process* at time t, that is

$$S_t^{(\delta)} = \sum_{j=0}^d \delta_t^j\, S_t^{(j)} \tag{2.8}$$

for $t \in [0, T]$. A strategy δ, or corresponding portfolio $S^{(\delta)}$, is said to be *self-financing* if

$$dS_t^{(\delta)} = \sum_{j=0}^d \delta_t^j\, dS_t^{(j)} \tag{2.9}$$

for $t \in [0,T]$. This means that no inflow or outflow of funds takes place for a corresponding self-financing portfolio and all changes in value are due to gains from trade in the primary security accounts. Since we deal only with self-financing strategies and portfolios we will therefore omit the phrase "self-financing".

For a given strategy δ let $\pi^j_{\delta,t}$ denote the jth *fraction* or *proportion* of the value of a corresponding strictly positive portfolio $S^{(\delta)}$ that is invested at time t in the jth primary security account, that is

$$\pi^j_{\delta,t} = \delta^j_t \, \frac{S^{(j)}_t}{S^{(\delta)}_t} \tag{2.10}$$

for $t \in [0,T]$ and $j \in \{0,1,\dots,d\}$. Note that the fractions always sum to unity such that

$$\sum_{j=0}^{d} \pi^j_{\delta,t} = 1 \tag{2.11}$$

for all $t \in [0,T]$. For a given strictly positive portfolio process $S^{(\delta)} = \{S^{(\delta)}_t,\, t \in [0,T]\}$ we obtain from (2.9), (2.7) and (2.10) the SDE

$$dS^{(\delta)}_t = S^{(\delta)}_t \left(r_t\, dt + \sum_{k=1}^{d} \sum_{j=1}^{d} \pi^j_{\delta,t}\, b^{j,k}_t \left(\theta^k_t\, dt + dW^k_t\right) \right) \tag{2.12}$$

for $t \in [0,T]$.

2.3 Growth Optimal Portfolio

We now identify the central object of our study, the *growth optimal portfolio* (GOP). From (2.12) it follows by application of the Itô formula that the logarithm of a strictly positive portfolio $S^{(\delta)}_t$ satisfies the SDE

$$d\ln(S^{(\delta)}_t) = g^\delta_t\, dt + \sum_{k=1}^{d} \sum_{j=1}^{d} \pi^j_{\delta,t}\, b^{j,k}_t\, dW^k_t \tag{2.13}$$

with *growth rate*

$$g^\delta_t = r_t + \sum_{k=1}^{d} \sum_{j=1}^{d} \pi^j_{\delta,t}\, b^{j,k}_t \left(\theta^k_t - \frac{1}{2} \sum_{j=1}^{d} \pi^j_{\delta,t}\, b^{j,k}_t \right) \tag{2.14}$$

for $t \in [0,T]$. The GOP arises naturally from maximizing the growth rate for all strictly positive portfolios.

Definition 2.2 *A GOP is a strictly positive portfolio process $S^{(\delta_*)}_t = \{S^{(\delta_*)}_t,\, t \in [0,T]\}$ with* maximum growth rate $g^{\delta_*}_t$, *such that*

$$g^{\delta_*}_t \geq g^\delta_t \tag{2.15}$$

for all $t \in [0,T]$ and all strictly positive portfolio processes $S^{(\delta)}$.

Note that the above definition of a GOP is given in a pathwise sense and does not use any conditional expectation or utility function. However, it is well-known that the portfolio which maximizes expected logarithmic utility from terminal wealth is the GOP, see Karatzas & Shreve (1998). Note that a GOP is unique up to the choice of its initial value.

To identify a GOP we have to maximize the growth rate, which is of quadratic form, with respect to the fractions given in (2.14). For each $t \in [0, T]$ this optimization problem is known to have a unique solution that can be identified by the first order conditions

$$0 = \sum_{k=1}^{d} b_t^{j,k} \left(\theta_t^k - \sum_{\ell=1}^{d} \pi_{\delta,t}^{\ell} \, b_t^{\ell,k} \right) \tag{2.16}$$

for each $t \in [0, T]$ and $j \in \{1, 2, \ldots, d\}$. The solution $\pi_{\delta,t} = (\pi_{\delta,t}^1, \ldots, \pi_{\delta,t}^d)^\top$ satisfies the equation

$$\pi_{\delta,t}^\top \, b_t = \theta_t^\top \tag{2.17}$$

for $t \in [0, T]$. Since by Assumption 2.1 the volatility matrix b_t is invertible for each $t \in [0, T]$, we can explicitly solve (2.17). Thus, a portfolio $S^{(\delta_*)}$ exists that achieves the maximum growth rate. This portfolio is a GOP, which is characterized by the vector of optimal fractions

$$\pi_{\delta_*,t} = \left(b_t^{-1} \right)^\top \, \theta_t \tag{2.18}$$

for all $t \in [0, T]$. The GOP value $S_t^{(\delta_*)}$ at time t satisfies by (2.18) and (2.12) the SDE

$$dS_t^{(\delta_*)} = S_t^{(\delta_*)} \left(\left[r_t + \sum_{k=1}^{d} (\theta_t^k)^2 \right] dt + \sum_{k=1}^{d} \theta_t^k \, dW_t^k \right) \tag{2.19}$$

for $t \in [0, T]$, where $S_0^{(\delta_*)} > 0$.

We call a continuous time financial market model of the above type a *continuous benchmark model*. We impose extremely weak conditions on a continuous benchmark model. The invertibility of the volatility matrix in Assumption 2.1 secures the existence of the GOP, but is not necessary as long as a solution to (2.5) exists. The existence of an equivalent risk neutral measure, as needed under the risk neutral approach, is *not* required. Therefore, in comparison with the risk neutral approach, the assumptions of the above benchmark model are less demanding.

In the above continuous benchmark model the short rate, volatility and market price for risk processes are very flexible, which provides substantial freedom in modeling. They may depend on nontraded sources of uncertainty. For instance, they do not require Markovianity of the dynamics. This generality does not impact on the structure of the solution of the above maximization problem, which identifies *locally in time* the optimal fractions of the GOP. In the following we will select similarly locally in time the fractions of a more general class of portfolios.

3 Portfolio Optimization

3.1 Portfolio Selection Theorem

Within this section we investigate what strategy a rational investor should naturally select to optimize the evolution of his or her wealth. Let us consider the *discounted portfolio* value

$$\bar{S}_t^{(\delta)} = \frac{S_t^{(\delta)}}{S_t^{(0)}}. \tag{3.1}$$

By (2.12) and application of the Itô formula we obtain from (3.1) the SDE

$$\begin{aligned} d\bar{S}_t^{(\delta)} &= \bar{S}_t^{(\delta)}\, \pi_{\delta,t}^\top\, b_t\, \{\theta_t\, dt + dW_t\} \\ &= \psi_{\delta,t}^\top\, \{\theta_t\, dt + dW_t\}, \end{aligned} \tag{3.2}$$

with *diffusion coefficients*

$$\psi_{\delta,t}^\top = (\psi_{\delta,t}^1, \dots, \psi_{\delta,t}^d) = \bar{S}_t^{(\delta)}\, \pi_{\delta,t}^\top\, b_t \tag{3.3}$$

for $t \in [0,T]$. According to (3.3) the discounted portfolio process $\bar{S}^{(\delta)}$ has the *discounted drift*

$$\alpha_t^\delta = \psi_{\delta,t}^\top\, \theta_t \tag{3.4}$$

and the *aggregate diffusion coefficient*

$$\gamma_t^\delta = \sqrt{\psi_{\delta,t}^\top\, \psi_{\delta,t}} \tag{3.5}$$

for $t \in [0,T]$.

Let us now identify a class of portfolios that we will call *optimal.* More precisely, a discounted portfolio will be called *optimal* if it exhibits for each time instant the largest discounted drift in comparison to all other portfolios with the same aggregate diffusion coefficient.

Definition 3.1 *A strictly positive portfolio process $S^{(\tilde{\delta})}$ is called* optimal *if for all $t \in [0,T]$ and all portfolios $S^{(\delta)}$ with a given aggregate diffusion coefficient value*

$$\gamma_t^{\tilde{\delta}} = \gamma_t^\delta \tag{3.6}$$

one has

$$\alpha_t^{\tilde{\delta}} \geq \alpha_t^\delta. \tag{3.7}$$

This optimality criterion is defined locally in time, avoids the introduction of a utility function and a time horizon. As we will see, it enables us to model in a flexible manner the effects that expected utility maximization aims to achieve.

As will become clear below, by choosing an optimal discounted portfolio the investor firstly decides on the level of risk, then he or she selects the portfolio with maximum discounted drift from within the class of all such discounted portfolios. Therefore, everything else being equal the investor prefers always more rather than less. This behavior of an investor is also termed *nonsatiation*. Note, for an optimal portfolio the initial value is a flexible parameter.

By the same methods that we use below one can show that for zero market prices for risk all portfolios are optimal. To avoid such unrealistic cases, we make the following assumptions.

Assumption 3.2 *The* total market price for risk

$$|\theta_t| = \sqrt{\theta_t^\top \theta_t} > 0 \tag{3.8}$$

is almost surely strictly positive and the fraction

$$\pi^0_{\delta_*,t} = 1 - \theta_t^\top b_t^{-1} \mathbf{1} \neq 1 \tag{3.9}$$

of wealth invested by the GOP in the savings account does not equal one for all $t \in [0, T]$.

The following *portfolio selection theorem* reveals the general structure of optimal portfolios.

Theorem 3.3 *The family of optimal portfolios* $S^{(\delta)}$ *is at time* $t \in [0, T]$ *parameterized by the fraction* $\pi^0_{\delta,t}$ *invested in the savings account. The discounted value* $\bar{S}^{(\delta)}_t$ *at time* t *of an optimal portfolio satisfies the SDE*

$$d\bar{S}^{(\delta)}_t = \bar{S}^{(\delta)}_t \frac{1 - \pi^0_{\delta,t}}{1 - \pi^0_{\delta_*,t}} \theta_t^\top (\theta_t \, dt + dW_t) \tag{3.10}$$

and the optimal fractions are of the form

$$\pi_{\delta,t} = \frac{1 - \pi^0_{\delta,t}}{1 - \pi^0_{\delta_*,t}} \left(b_t^{-1}\right)^\top \theta_t \tag{3.11}$$

for $t \in [0, T]$.

The fact that the solution to the given constrained optimization problem depends on $\pi^0_{\delta,t}$ is interesting and not evident ahead. As we will see, mathematically the above results require no more than multivariate calculus and a basic understanding of Itô calculus. The proof of Theorem 3.3 is given in Appendix A. It uses similar arguments as that given in Platen (2002) or Khanna & Kulldorff (1999), who treated the special case with deterministic coefficients. By (3.11) and (2.18) it follows that any investor who prefers an optimal portfolio follows a strategy where at each time a fraction of wealth is invested in the GOP and the rest is held in the savings account. It is very important to note that an optimal portfolio retains a free parameter, which can be specified as the fraction of wealth to be held in the savings account.

This is a consequence of our definition of optimality, where we maximize the entire discounted drift of a portfolio while keeping the aggregate diffusion coefficient value fixed. The GOP can be interpreted as a *mutual fund* since it is part of all optimal portfolios. Each optimal portfolio can be obtained by a, in general, randomly changing combination of the mutual fund and the savings account. This shows that the GOP plays a natural role in portfolio selection.

Note that we obtained the above form of optimal fractions in (3.11) without exploiting any Markovianity, expected utility or equilibrium argument. If one wants to enter utility into the modeling, then it would have to determine the ratio of the value of risky assets in the optimal portfolio to the total optimal portfolio value. Mainly due to its generality the above mutual fund or portfolio selection theorem is different to most other portfolio selection theorems presented in the literature, see, for instance, Merton (1971) or Karatzas & Shreve (1998). As we will see, in some cases expected utility maximization can be embedded in the framework of optimal portfolios.

3.2 Risk Aversion Coefficient

For an optimal portfolio $S^{(\delta)}$ one can say that the fraction $\pi^0_{\delta,t}$ invested in the savings account reflects the risk aversion of the investor. Similar to the Arrow-Pratt absolute risk aversion coefficient, see Luenberger (1997), we can now introduce for an optimal portfolio $S^{(\delta)}$ the *risk aversion coefficient*

$$J_\delta(t, \bar{S}^{(\delta)}_t) = \frac{1 - \pi^0_{\delta_*,t}}{1 - \pi^0_{\delta,t}} \tag{3.12}$$

at time $t \in [0, T]$. According to Theorem 3.3, an investor who forms an optimal portfolio invests some fraction of his or her wealth into the GOP and allocates the remaining part to the savings account. For instance, if all of the investor's wealth is invested in the savings account, then he or she is completely risk averse and the risk aversion coefficient is by (3.12) not finite. Note that the risk aversion coefficient depends not only on time but also on the level of discounted wealth that is invested.

For an optimal portfolio $S^{(\delta)}$ the vector of fractions in the risky assets can be written by (3.11) and (3.12) in the form

$$\pi_{\delta,t} = \frac{\pi_{\delta_*,t}}{J_\delta(t, \bar{S}^{(\delta)}_t)} \tag{3.13}$$

for $t \in [0,T]$. In the special case when an investor maximizes the growth rate of an optimal portfolio, then he or she obtains a GOP and has by (3.12) a risk aversion coefficient of value one, that is,

$$J_{\delta_*}(t, \bar{S}_t^{(\delta_*)}) = 1 \tag{3.14}$$

for all $t \in [0,T]$.

The risk aversion coefficient appears to be useful as a flexible *parameter process* for modeling the evolution of the risk aversion of an investor over time. One can let this parameter process depend on a wide range of factors, not only the investor's level of wealth. For instance, one can make it dependent on changing life circumstances, government policies, taxation, herd behavior or other social or financial factors. The above proposed risk aversion coefficient provides substantial flexibility for modeling and can be easily applied to multi-factor models. In the following we will consider an example with utility functions. These are usually difficult to handle computationally in models with more than two factors.

3.3 Utility Maximization

Let us illustrate a standard situation in a Markovian framework, where the risk aversion coefficient (3.12) relates to expected utility maximization. In the simplest case one may assume that b_t and θ_t are deterministic. Consider a strictly increasing and strictly concave, twice differentiable utility function $U(\cdot)$ defined on $(0,\infty)$ for all $t \in [0,T]$. We fix the planning horizon at the deterministic terminal time T and endow an investor with the discounted investable wealth $x > 0$ at time $t \in [0,T]$. By $\mathcal{V}_t$ we denote the set of fraction processes $\pi_\delta = \{\pi_{\delta,s},\ s \in [t,T]\}$ of nonnegative portfolios for the investment in risky primary security accounts. The investor is assumed to form a nonnegative portfolio $S^{(\delta)}$ with $\pi_\delta \in \mathcal{V}_t$. The expected utility

$$E\left(U(\bar{S}_T^{(\delta)}) \,\Big|\, \bar{S}_t^{(\delta)} = x\right)$$

is maximized with respect to the choice of different fraction processes of nonnegative portfolios with $\bar{S}_t^{(\delta)} = x$. Let us assume that we are given the *value function*

$$u(t,x) = \sup_{\pi_\delta \in \mathcal{V}_t} E\left(U(\bar{S}_T^{(\delta)}) \,\Big|\, \bar{S}_t^{(\delta)} = x\right) \tag{3.15}$$

of this problem for $(t,x) \in [0,T] \times [0,\infty)$. As we will see, the value function $u(t,x)$ can be interpreted as a kind of utility function itself. For a wide class of utility functions and dynamics it has been shown, see, for instance Krylov (1980), Cox & Huang (1989), Fleming & Soner (1992), Korn (1997) and Karatzas & Shreve (1998), that the partial derivatives $\frac{\partial u}{\partial t}$, $\frac{\partial u}{\partial x}$ and $\frac{\partial^2 u}{\partial x^2}$ are defined and continuous on $[0,T) \times (0,\infty)$. Furthermore, on the basis of the, so called, Bellman principle, see Fleming & Soner (1992), it has been demonstrated for a wide range of diffusion dynamics that the value function $u(\cdot,\cdot)$ satisfies the *Hamilton-Jacobi-Bellman equation*

$$\frac{\partial u(t,x)}{\partial t} + \sup_{\pi_\delta \in \mathcal{V}_t} \left(\pi_{\delta,t}^\top \, b_t \, \theta_t \, x \, \frac{\partial u(t,x)}{\partial x} + \frac{1}{2} \, \pi_{\delta,t}^\top \, b_t \, b_t^\top \, \pi_{\delta,t} \, x^2 \, \frac{\partial^2 u(t,x)}{\partial x^2} \right) = 0 \qquad (3.16)$$

for $t \in [0,T)$, $x \in (0,\infty)$ with *terminal condition*

$$u(T,x) = U(x) \qquad (3.17)$$

for $x \in (0,\infty)$. To achieve in (3.16) a genuine maximum it is necessary that the fractions satisfy the first order conditions

$$\frac{\partial}{\partial \pi_{\delta,t}^j} \left[\sum_{k=1}^d \sum_{\ell=1}^d \pi_{\delta,t}^\ell \, b_t^{\ell,k} \, \theta_t^k \, x \, \frac{\partial u(t,x)}{\partial x} + \frac{1}{2} \sum_{k=1}^d \left(\sum_{\ell=1}^d \pi_{\delta,t}^\ell \, b_t^{\ell,k} \right)^2 x^2 \, \frac{\partial^2 u(t,x)}{\partial x^2} \right] = 0 \qquad (3.18)$$

for all $t \in [0,T]$ and $j \in \{1,2,\ldots,d\}$. This leads to a solution that satisfies, when expressed in vector and matrix form, the vector equation

$$b_t \, x \left[\theta_t \, \frac{\partial u(t,x)}{\partial x} + b_t^\top \, \pi_{\delta,t} \, x \, \frac{\partial^2 u(t,x)}{\partial x^2} \right] = (0,\ldots,0)^\top. \qquad (3.19)$$

If one substitutes the fractions of an optimal portfolio, see (3.11) and (3.12), into (3.19), then one notes for the case $J_\delta(t, \bar{S}_t^{(\delta)}) \neq 0$ that the condition (3.19) becomes

$$b_t \, x \, \theta_t \left[\frac{\partial u(t,x)}{\partial x} + \frac{x}{J_\delta(t,x)} \, \frac{\partial^2 u(t,x)}{\partial x^2} \right] = (0,\ldots,0)^\top. \qquad (3.20)$$

It follows that an optimal portfolio maximizes the expected utility in (3.15), where the risk aversion coefficient is of the form

$$J_\delta(t,x) = -x \, \frac{\frac{\partial^2 u(t,x)}{\partial x^2}}{\frac{\partial u(t,x)}{\partial x}}. \qquad (3.21)$$

This particular form of the risk aversion coefficient resembles the Arrow-Pratt absolute risk aversion coefficient, see Luenberger (1997), when interpreting $u(t,\cdot)$ as a utility function itself. In the typical case when $u(t,x)$ is strictly increasing and strictly concave in x, the risk aversion coefficient is strictly positive.

By (3.21) and (3.16) we get for the value function the first order partial differential equation (PDE)

$$\frac{\partial u(t,x)}{\partial t} + \frac{1}{2} \, \frac{|\theta_t|^2 \, x}{J_\delta(t,x)} \, \frac{\partial u(t,x)}{\partial x} = 0 \qquad (3.22)$$

for $t \in [0,T]$, $x \in (0,\infty)$. Alternatively, we have by (3.22) and (3.21) the PDE

$$\frac{\partial u(t,x)}{\partial t} - \frac{1}{2} \left(\frac{|\theta_t| \, x}{J_\delta(t,x)} \right)^2 \frac{\partial^2 u(t,x)}{\partial x^2} = 0 \qquad (3.23)$$

for $t \in [0,T]$, $x \in (0,\infty)$.

Note that one can generalize this example to include production and consumption, as well as time dependence and time integrals over discounted utility functions. The maximization of a discounted and integrated expected utility for an infinite time horizon can be similarly treated.

To illustrate more directly the link between a given utility function and the corresponding risk aversion coefficient, consider the classical Merton problem when using power utility. In Merton (1971) an explicit solution for asset price dynamics under deterministic market prices for risk, short rate and volatilities, when using power utility $U(x) = \frac{1}{\gamma}\, x^\gamma$, was given with

$$u(t, \bar{S}_t^{(\delta)}) = \exp\left\{ - \int_t^T \frac{|\theta_s|^2\, \gamma}{2\,(1-\gamma)} ds \right\} \left(\bar{S}_t^{(\delta)} \right)^\gamma \tag{3.24}$$

for $t \in [0,T]$, $\gamma \in (0,1)$. The resulting vector of optimal fractions takes the form

$$\pi_{\delta,t} = \left(b_t^{-1} \right)^\top \theta_t \, \frac{1}{1-\gamma}. \tag{3.25}$$

One notes by (3.11) for an optimal portfolio $S^{(\delta)}$ the risk aversion coefficient in this case equals the constant

$$J_\delta(t, \bar{S}_t^{(\delta)}) = 1 - \gamma \tag{3.26}$$

for all $t \in [0,T]$. This means that power utility yields a constant risk aversion coefficient.

One notes from (3.26) that for $\gamma \to 0$ the risk aversion coefficient approaches one. This corresponds to the choice of the GOP as optimal portfolio. Indeed, one can show for logarithmic utility $U(x) = \ln(x)$ that the value function is

$$u(t, \bar{S}_t^{(\delta)}) = \ln\left(\bar{S}_t^{(\delta)} \right) + \int_t^T \frac{|\theta_s|^2}{2}\, ds \tag{3.27}$$

for $t \in [0,T]$. The corresponding vector of optimal fractions is of the form

$$\pi_{\delta,t} = \left(b_t^{-1} \right)^\top \theta_t \tag{3.28}$$

for $t \in [0,T]$, as expected, see (2.18). On the other hand, for $\gamma \to 1$ the risk aversion coefficient approaches infinity, which corresponds to the choice of the savings account as optimal portfolio.

It is clear from (3.21), when considering explicitly given or numerically obtained value functions, then one can directly determine the corresponding risk aversion coefficient in dependence on the given time and actual value of the discounted portfolio. As shown in Fleming & Soner (1992), Karatzas & Shreve (1998) and Korn (1997) it is, in general, difficult to compute the corresponding value function for a given utility function and specific market dynamics. The above proposed concept of optimal portfolios with risk aversion coefficient provides a method of circumventing these problems.

It allows the modeling of optimal portfolios for different investors with a wide range of risk aversion coefficients. Obviously, the concepts of an optimal portfolio and a risk aversion coefficient remove in investment planning the necessity to fix a particular time horizon and a particular utility function. In more general multi-factor portfolio selection problems than the standard situation described above, it is practically impossible to obtain even numerically a solution of the expected utility maximization problem because multi-dimensional partial differential equations are involved. However, it is still straightforward in such a case to determine the class of optimal portfolios and select one by characterizing the risk aversion coefficient process.

3.4 Market Portfolio as Optimal Portfolio

The total investable wealth of the ℓth investor is denoted by $S^{(\delta_\ell)}$, $\ell \in \{1, 2, \ldots, n\}$. The *market portfolio* $S^{(\delta_+)}$ of investable wealth is then the sum of the total investable wealth processes of all investors. In the simplest case an investor may participate in a pension fund which could be expected to form an optimal portfolio. Let us assume that each investor holds an optimal portfolio.

Assumption 3.4 *Each investor forms an optimal portfolio with his or her total investable wealth.*

Therefore, we get from (3.10) for the discounted value $\bar{S}_t^{(\delta_+)}$ of the market portfolio at time t the SDE

$$\begin{aligned} d\bar{S}_t^{(\delta_+)} &= d\left(\sum_{\ell=1}^{n} \bar{S}_t^{(\delta_\ell)}\right) \\ &= \sum_{\ell=1}^{n} \bar{S}_t^{(\delta_\ell)} \, \frac{1-\pi^0_{\delta_\ell,t}}{1-\pi^0_{\delta_*,t}} \, \theta_t^\top \, (\theta_t \, dt + dW_t) \\ &= \sum_{\ell=1}^{n} \left(\bar{S}_t^{(\delta_\ell)} - \delta_\ell^0(t)\right) \, \frac{\theta_t^\top}{1-\pi^0_{\delta_*,t}} \, (\theta_t \, dt + dW_t) \\ &= \bar{S}_t^{(\delta_+)} \, \frac{1-\pi^0_{\delta_+,t}}{1-\pi^0_{\delta_*,t}} \, \theta_t^\top \, (\theta_t \, dt + dW_t) \qquad (3.29) \end{aligned}$$

for $t \in [0, T]$. One obtains from (3.29), Definition 3.1 and the SDE (3.10) the following result.

Corollary 3.5 *The market portfolio of investable wealth is an optimal portfolio. It forms a GOP if and only if*

$$\pi^0_{\delta_+,t} = \pi^0_{\delta_*,t} \tag{3.30}$$

for all $t \in [0, T]$.

By (3.12) the risk aversion coefficient of the market portfolio equals

$$J_{\delta_+}(t, \bar{S}^{(\delta_+)}_t) = \frac{1 - \pi^0_{\delta_*,t}}{1 - \pi^0_{\delta_+,t}} \tag{3.31}$$

for all $t \in [0, T]$. Note that if the market portfolio is not the GOP, then it is still always a combination of the GOP and the savings account.

Let us now discuss consequences of the hypothetical assumption that the monetary authorities aim to maximize the growth rate of the market portfolio of investable wealth. One could potentially argue that they may be able to achieve this goal by adjusting the short rate or equivalently the short term money supply. Obviously, the maximization of the growth rate of the market portfolio is equivalent to the selection of the GOP as market portfolio of investable wealth. This refers to the choice of a risk aversion coefficient with constant value one, see (3.14). This hypothetical assumption would lead directly to the observability of the GOP in form of the market portfolio of investable wealth. We remark, that in Platen (2004a) it is assumed that the savings account is in net zero supply, which makes the market portfolio equal to the GOP if $\pi^0_{\delta_*,t} = 0$ for all $t \in [0, T]$.

Alternatively, the above indicated proximity of market portfolio and GOP results in an asymptotic sense via a limit theorem derived in Platen (2004b, 2004c), where sequences of benchmark models for an increasing number d of risky primary security accounts have been studied. It has been shown that sequences of globally diversified portfolios with fractions that vanish sufficiently fast as $d \to \infty$, converge asymptotically towards the GOP. Therefore, if the market portfolio of investable wealth can be considered to be diversified, then one has a robustness property in the sense that all diversified portfolios, including the GOP, are asymptotically very similar.

3.5 Market Prices for Risk and Risk Premia

In relation (2.5) we introduced the market price for risk vector $\theta_t = (\theta^1_t, \theta^2_t, \ldots, \theta^d_t)^\top$, which is given by (2.6). In practice, the estimation of the appreciation rate vector a_t, as a parameter process in the drift of a diffusion process, is not realistic. Simple statistical analysis of the estimates reveal that there is probably not enough market data available, covering a sufficiently long time period, that allows the direct estimation of parameters in the appreciation rate with a reasonable confidence level. However, by using equation (2.18) for the fractions of the GOP one can express the market price for risk vector in the form

$$\theta_t = b_t^\top \pi_{\delta_*,t} \tag{3.32}$$

for all times $t \in [0, T]$. That is, by (2.10) we obtain indirectly the kth market price for risk as an average of the kth volatilities of the primary security accounts weighted by the values invested in the GOP, that is

$$\theta_t^k = \frac{\sum_{j=0}^d b_t^{j,k}\, \delta_*^j(t)\, S_t^{(j)}}{\sum_{i=0}^d \delta_*^i(t)\, S_t^{(i)}} \tag{3.33}$$

for $t \in [0, T]$ and $k \in \{1, 2, \ldots, d\}$. By (3.32) and (3.33) one notes that market prices for risk are averages of volatilities.

By (2.6) one obtains the risk premium vector $p(t) = a_t - r_t\, \mathbf{1} = b_t\, \theta_t$ of risky primary security accounts by using the representation

$$p(t) = b_t\, b_t^\top \pi_{\delta_*,t} \tag{3.34}$$

for all $t \in [0, T]$. Consequently, risk premia are averages of products of volatilities.

By the interpretation of the GOP as market portfolio of investable wealth one gains an alternative access to the estimation of risk premia and market prices for risk. In this case one can observe also the fractions of primary security accounts from the known market capitalization. Additionally, one can estimate the volatility matrix from frequently observed primary security account data, which is not a simple task. However, it appears to be simpler than estimating appreciation rates, growth rates or risk premia. For this purpose it is reasonable to use, for instance, quadratic variations of logarithms of security prices. Since these quantities can be directly inferred from the data of a relatively short observation period, one can obtain a reasonable estimate for the market price for risk from estimated quantities on the right hand side of equation (3.32). This also indicates a potential route for estimating risk premia, see (3.34), which provides a new method for studying the *equity premium*, see Mehra & Prescott (1985).

3.6 Risk Measurement

Important regulatory requirements for risk measurement, see Basle (1996), ask for using a *broadly based index*, as discussed in Platen & Stahl (2003), for the measurement of market risk. As we will see, the GOP is again the financial quantity that, when used as reference unit or benchmark, provides a transparent and well structured description.

Let us now interpret the GOP as a broadly diversified index, which is obtained by a linear combination of the market portfolio and the savings account. Then the volatility of the GOP

$$|\theta_t| = \sqrt{\sum_{k=1}^{d} (\theta_t^k)^2}, \tag{3.35}$$

which equals the *total market price for risk*, models in a natural way the *general market risk*, also termed *systematic risk*. Let us refer to values that are expressed in units of the GOP, as *benchmarked* values. The jth benchmarked primary security account process $\hat{S}^{(j)} = \{\hat{S}_t^{(j)},\ t \in [0,T]\}$, with

$$\hat{S}_t^{(j)} = \frac{S_t^{(j)}}{S_t^{(\delta_*)}}, \tag{3.36}$$

satisfies, by (2.7), (2.19) and application of the Itô formula, the driftless SDE

$$d\hat{S}_t^{(j)} = -\hat{S}_t^{(j)} \sum_{k=1}^{d} \sigma_t^{j,k}\, dW_t^k \tag{3.37}$$

for $t \in [0,T]$ and $j \in \{0, 1, \ldots, d\}$. Here, the (j,k)th *specific volatility*

$$\sigma_t^{j,k} = \theta_t^k - b_t^{j,k} \tag{3.38}$$

of the benchmarked jth primary security account $\hat{S}^{(j)}$ arises from an application of the Itô formula and measures the jth *specific market risk* at time $t \in [0,T]$ with respect to the kth Wiener process W^k for $k \in \{1, 2, \ldots, d\}$ and $j \in \{0, 1, \ldots, d\}$. The above separation of general and specific market risk is a natural feature of the benchmark approach.

For the modeling of *all* primary security accounts it is only necessary to specify the volatility processes $\sigma^{j,k}$ for all $j \in \{0, 1, \ldots, d\}$ and $k \in \{1, 2, \ldots, d\}$ together with the short rate process r and appropriate initial values. To see this, note that by (3.36) we obtain the GOP from the benchmarked savings account $\hat{S}^{(0)}$ and the savings account $S^{(0)}$ as the ratio

$$S_t^{(\delta_*)} = \frac{S_t^{(0)}}{\hat{S}_t^{(0)}} \tag{3.39}$$

for all $t \in [0,T]$. From the jth benchmarked primary security account $\hat{S}^{(j)}$ and the GOP $S^{(\delta_*)}$ we can then derive by (3.36) the value of the jth primary security account in the form

$$S_t^{(j)} = \hat{S}_t^{(j)}\, S_t^{(\delta_*)} \tag{3.40}$$

for $t \in [0,T]$ and $j \in \{1, 2, \ldots, d\}$.

In the case when one models the market from the perspective of the domestic currency, as is the case in this paper, the general market risk is reflected by the market price for risk processes $\sigma^{0,k} = \theta^k$, $k \in \{1, 2, \ldots, d\}$. If the first primary security account $S^{(1)}$ were a foreign savings account, then the volatility processes $\sigma^{1,k}$, $k \in \{1, 2, \ldots, d\}$, would provide the market prices for risk for this foreign currency denomination.

3.7 Capital Asset Pricing Model

Sharpe (1964), Lintner (1965), Mossin (1966) and Merton (1973a) developed the seminal *capital asset pricing model* (CAPM) as an equilibrium model. As we will see, one does not need an equilibrium or expected utility argument to obtain the core statement of the CAPM. To demonstrate this, let us introduce the *risk premium* $p_\delta(t)$ at time $t \in [0,T]$ of a portfolio $S^{(\delta)}$ as the expected excess return, which means by (2.12) and (2.19) that

$$p_\delta(t) = \sum_{k=1}^{d} \sum_{j=1}^{d} \pi^j_{\delta,t}\, b^{j,k}_t\, \theta^k_t \tag{3.41}$$

for $t \in [0,T]$.

Let $\langle X, Y\rangle_t$ denote the covariation of two processes X and Y at time t, which is defined as the limit in probability of the sum of the products of the increments of X and Y based on a time discretization with vanishing step size, see Karatzas & Shreve (1991).

The *systematic risk parameter* $\beta_\delta(t)$ of a portfolio $S^{(\delta)}$, the *beta*, can then be defined as

$$\beta_\delta(t) = \frac{\frac{d}{dt}\langle \ln(S^{(\delta)}), \ln(S^{(\delta_+)})\rangle_t}{\frac{d}{dt}\langle \ln(S^{(\delta_+)}), \ln(S^{(\delta_+)})\rangle_t} \tag{3.42}$$

for $t \in [0,T]$, where $S^{(\delta_+)}$ denotes the market portfolio of investable wealth. Under Assumption 3.4 we have shown that the market portfolio is an optimal portfolio. Therefore, it follows by (3.42), (3.29), (3.12) and (2.12) that

$$\begin{aligned} \beta_\delta(t) &= \frac{\sum_{k=1}^{d}\sum_{j=1}^{d} \pi^j_{\delta,t}\, b^{j,k}_t\, \theta^k_t\, \frac{1-\pi^0_{\delta_+,t}}{1-\pi^0_{\delta_*,t}}}{\left(\frac{1-\pi^0_{\delta_+,t}}{1-\pi^0_{\delta_*,t}}\right)^2 |\theta_t|^2} \\ &= \frac{J_\delta(t, \bar{S}^{(\delta_+)})}{|\theta_t|^2} \sum_{k=1}^{d}\sum_{j=1}^{d} \pi^j_{\delta,t}\, b^{j,k}_t\, \theta^k_t. \end{aligned} \tag{3.43}$$

This yields by (3.41) and (3.29) the following conclusion.

Corollary 3.6 *The systematic risk parameter of a portfolio $S^{(\delta)}$ has the form*

$$\beta_\delta(t) = \frac{p_\delta(t)}{p_{\delta_+}(t)} \tag{3.44}$$

for $t \in [0,T]$.

The portfolio beta in (3.44) is exactly in the form that the CAPM theoretically suggests. This leads directly to the core statement of the CAPM without relying on any equilibrium or expected utility argument.

3.8 Efficient Frontier and Efficient Growth Rate

Let us now derive the *Markowitz efficient frontier*, see Markowitz (1952, 1959), in the given continuous benchmark model. For an optimal portfolio $S^{(\delta)}$ it follows by (3.10) that its *aggregate squared volatility* equals

$$|b_\delta(t)|^2 = \left(\frac{1-\pi^0_{\delta,t}}{1-\pi^0_{\delta_*,t}}\right)^2 |\theta_t|^2 = \frac{|\theta_t|^2}{\left(J_\delta(t,\bar{S}^{(\delta)}_t)\right)^2} \tag{3.45}$$

and its risk premium takes the form

$$p_\delta(t) = \frac{1-\pi^0_{\delta,t}}{1-\pi^0_{\delta_*,t}}\,|\theta_t|^2 = |b_\delta(t)|\,|\theta_t| \tag{3.46}$$

for $t \in [0,T]$, see (3.41). Obviously, we have $p_\delta(t) \geq 0$, such that by (3.10), (3.45) and (3.46) the *appreciation rate* $a_\delta(t)$ for an optimal portfolio can be written as a function of its squared volatility $|b_\delta(t)|^2$ in the form

$$a_\delta(t) = r_t + p_\delta(t) = r_t + |b_\delta(t)|\,|\theta_t| = r_t + \sqrt{|b_\delta(t)|^2}\,|\theta_t| \tag{3.47}$$

for $t \in [0,T]$. This function can be interpreted as the well-known *Markowitz efficient frontier*. More precisely, each optimal portfolio $S^{(\delta)}$ has an appreciation rate that is located at the efficient frontier $a_\delta(t)$ given in (3.47). Note that the efficient frontier moves stochastically up and down over time in dependence on the short rate r_t. Its slopes change also over time according to the, generally, stochastic total market price for risk $|\theta_t|$. For a given time instant $t \in [0,T]$ the Figure 1 shows the efficient frontier in dependence on the squared volatility

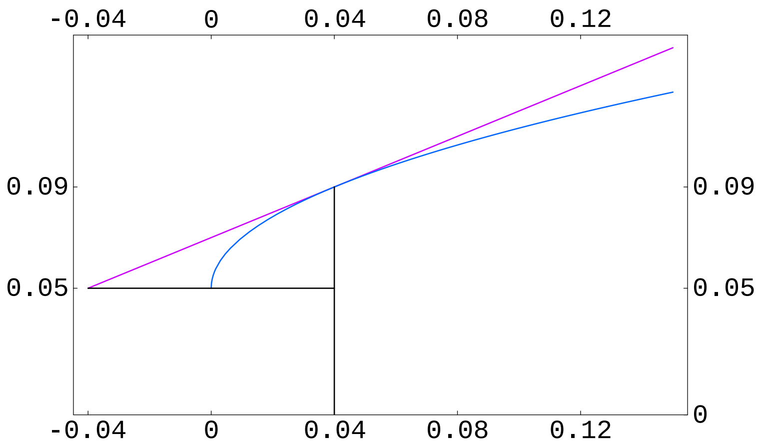

Figure 1: Efficient frontier.

$|b_\delta(t)|^2$ of the optimal portfolios, where $r_t = 0.05$ and $|\theta_t|^2 = 0.04$. This graph also includes the tangent with slope $\frac{1}{2}$ at the point $|b_\delta(t)|^2 = |\theta_t|^2$, which corresponds to the squared volatility of the GOP. The second equation in (3.47) describes the

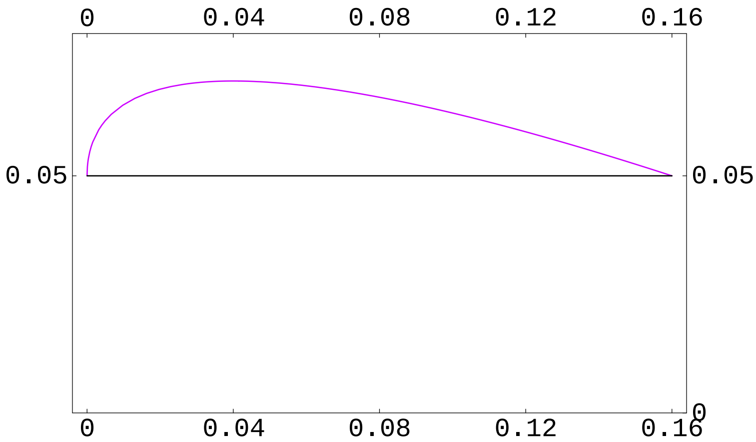

Figure 2: Efficient growth rates.

well-known *capital market line* in dependence on the portfolio volatility $|b_\delta(t)|$, see Luenberger (1997). Here the slope is directly the market price for risk.

For illustration, we plot in Figure 2 for given $t \in [0, T]$ the growth rates of optimal portfolios $S^{(\delta)}$ in dependence on their squared volatility $|b_\delta(t)|^2$. One could call these growth rates the *efficient growth rates*, which have the form

$$g_\delta(t) = r_t + \sqrt{|b_\delta(t)|^2}\,|\theta_t| - \frac{1}{2}\,|b_\delta(t)|^2, \tag{3.48}$$

see (2.14), (3.11) and (3.45). One notes that for $|b_\delta(t)|^2 = |\theta_t|^2$ the growth rates achieve their maximum, which yields the growth rate of the GOP. For $|b_\delta(t)| = 2\,|\theta_t|$ the efficient growth rate equals the short rate. As we will see later, the GOP is the best performing portfolio under various criteria, in particular, for long term investors, see Luenberger (1997).

3.9 Sharpe Ratio

Another important quantity in modern portfolio theory is the *Sharpe ratio*, see Sharpe (1964). For any portfolio $S^{(\delta)}$ the Sharpe ratio $s_\delta(t)$ at time t is defined as the risk premium $p_\delta(t)$ over the aggregate volatility $|b_\delta(t)|$, that is,

$$s_\delta(t) = \frac{p_\delta(t)}{|b_\delta(t)|} \tag{3.49}$$

as long as $|b_\delta(t)| = \sqrt{\sum_{k=1}^d (\sum_{j=1}^d \pi^j_{\delta,t}\, b^{j,k}_t)^2} > 0,\ t \in [0, T]$. It follows by (3.3)–(3.5) that the Sharpe ratio at time t equals the ratio

$$s_\delta(t) = \frac{\alpha^\delta_t}{\gamma^\delta_t} \tag{3.50}$$

of the discounted drift over the aggregate diffusion coefficient. This simple but important observation allows us to employ Theorem 3.3, which identifies the maximum value of α_t^δ for given γ_t^δ when choosing an optimal portfolio with a given risk aversion coefficient. Due to the structure of the discounted portfolio drift and aggregate diffusion coefficient for any optimal portfolio $S^{(\delta)}$, as given in (3.10), the Sharpe ratio (3.50) equals by (3.3), (3.5) and (3.11) for all $t \in [0,T]$ the total market price for risk, that is

$$s_\delta(t) = |\theta_t|. \tag{3.51}$$

Note that the equality follows already directly from (A.6) in the Appendix. More generally, by (3.50) and Theorem 3.3 one can draw the following conclusion.

Corollary 3.7 *For any risky portfolio $S^{(\delta)}$ its Sharpe ratio $s_\delta(t)$ is never greater than the market price for risk, that is*

$$s_\delta(t) \leq |\theta_t| \tag{3.52}$$

for all $t \in [0,T]$.

This result is highly relevant for modern portfolio optimization. It limits the achievable Sharpe ratio for any risky portfolio. It is similar to the maximum growth rate attained by the GOP, see (2.14)–(2.15), which limits the achievable growth rates of all strictly positive portfolios. The optimal portfolios are basically those with maximum Sharpe ratio. The method employed for the derivation of Corollary 3.7 is similar to that described in Fleming & Stein (2004). If one asks for the portfolio that has the largest Sharpe ratio and also the largest growth rate, then one obtains the GOP. As demonstrated in this section the GOP is a very useful tool for the intertemporal generalization of the Markowitz-Tobin-Sharpe static mean variance portfolio analysis.

4 Fair Pricing

4.1 Benchmarked Portfolios

Similar to (3.36) we define for any portfolio $S^{(\delta)}$ its *benchmarked* value

$$\hat{S}_t^{(\delta)} = \frac{S_t^{(\delta)}}{S_t^{(\delta_*)}} \tag{4.1}$$

for $t \in [0,T]$. By application of the Itô formula together with (2.12) and (2.19) we obtain for $\hat{S}_t^{(\delta)}$ the driftless SDE

$$d\hat{S}_t^{(\delta)} = -\sum_{k=1}^{d}\sum_{j=0}^{d} \delta_t^{(j)}\, \hat{S}_t^{(j)}\, \sigma_t^{j,k}\, dW_t^k \tag{4.2}$$

for $t \in [0, T]$. Thus, by (4.2) any benchmarked nonnegative portfolio $\hat{S}^{(\delta)}$ is an $(\underline{\mathcal{A}}, P)$-*local martingale.* As shown in Rogers & Williams (2000), any nonnegative local martingale is a supermartingale. Thus, we obtain the following important result.

Corollary 4.1 *All nonnegative benchmarked portfolios are $(\underline{\mathcal{A}}, P)$-supermartingales.*

In the following we draw some conclusions from this statement, which do not require any major assumptions.

4.2 Arbitrage

In developed economies there exists the legal concept of *limited liability.* This means that any investor must own a strictly positive portfolio of total investable wealth. As in (3.29), the market portfolio of investable wealth can therefore be decomposed into the total investable wealth processes of all investors. If the total investable wealth process of an investor reaches the level zero or becomes negative, then he or she must declare bankruptcy, respectively. This is a crucial property of a market that is modeled by the following notion of *no-arbitrage*, see Platen (2002).

Definition 4.2 *A nonnegative portfolio $S^{(\delta)}$ is said to permit* arbitrage *if for $S_0^{(\delta)} = 0$, almost surely, we have*

$$P(S_T^{(\delta)} > 0) > 0. \tag{4.3}$$

Thus, in the case of arbitrage there exists a nonnegative portfolio process, which generates from zero initial capital, strictly positive wealth with strictly positive probability. Using the supermartingale property of benchmarked nonnegative portfolios, described in Corollary 4.1, we proof the following result.

Corollary 4.3 *A continuous benchmark model does not permit arbitrage.*

Proof: Consider a nonnegative portfolio process $S^{(\delta)}$, where we have

$$S_T^{(\delta)} \geq S_0^{(\delta)} = 0 \tag{4.4}$$

almost surely. By the supermartingale property of $\hat{S}^{(\delta)}$, see Corollary 4.1, we obtain

$$E\left(\hat{S}_T^{(\delta)}\right) = E\left(\hat{S}_T^{(\delta)} \mid \mathcal{A}_0\right) \leq \hat{S}_0^{(\delta)} = 0, \tag{4.5}$$

almost surely. Due to (4.4) and (4.5) the nonnegative benchmarked value $\hat{S}_T^{(\delta)}$ cannot be strictly greater than zero with any strictly positive probability, that is

$$P\left(\hat{S}_T^{(\delta)} > 0\right) = 0.$$

Thus, it follows $P(\hat{S}_T^{(\delta)} > 0) = 0$ and the inequality (4.3) cannot hold, which proves by Definition 4.2 with (4.3) that there is no arbitrage. □

4.3 Fair Contingent Claim Pricing

The described class of benchmark models is more general than the continuous market models admitted under the standard risk neutral approach as presented in Karatzas & Shreve (1998) or Björk (1998). The main difference is that an equivalent risk neutral martingale measure need not exist. Therefore, the standard risk neutral pricing methodology is, in general, not applicable. A consistent pricing methodology is needed that generalizes risk neutral pricing. Following Platen (2002) one can use the *fair pricing concept*. We call a value process *fair* if its benchmarked value forms an $(\underline{\mathcal{A}}, P)$-martingale. Note that a value process need not to be a portfolio process.

An $\mathcal{A}_\tau$-measurable random variable H_τ with

$$E\left(\frac{|H_\tau|}{S_\tau^{(\delta_*)}}\right) < \infty, \tag{4.6}$$

which pays the amount H_τ at some stopping time τ, is called a *contingent claim*. For a given contingent claim the corresponding *fair price* $U_{H_\tau}(t)$ at time $t \in [0, \tau]$ is uniquely determined by the *fair pricing formula*

$$U_{H_\tau}(t) = S_t^{(\delta_*)}\, E\left(\frac{H_\tau}{S_\tau^{(\delta_*)}} \,\middle|\, \mathcal{A}_t\right). \tag{4.7}$$

Note that the above fair pricing formula relies on the GOP, which is used as numeraire portfolio, see Long (1990). Note that *not* all benchmarked portfolios form $(\underline{\mathcal{A}}, P)$-martingales. By using the GOP as numeraire and the real world probability measure as pricing measure, the fair pricing concept provides a constructive and simple way of valuing contingent claims.

For fair pricing in practice it is advantageous to have a good proxy for the GOP as numeraire. Assumptions which ensure that the market portfolio of investable wealth is a good proxy for the GOP were described in Section 3.4. By modeling and calibrating the dynamics of the GOP it is possible to compute the real world expectation in (4.7).

4.4 Risk Neutral and Actuarial Pricing

Let us illustrate that the risk neutral pricing methodology appears as a particular case of fair pricing. In a continuous benchmark model a presumed risk neutral probability measure P_θ has the Radon-Nikodym derivative $\Lambda_T = \frac{dP_\theta}{dP}$ with $\Lambda = \{\Lambda_t,\, t \in [0,T]\}$, where

$$\Lambda_t = \frac{S_0^{(\delta_*)}}{S_t^{(\delta_*)}} \frac{S_t^{(0)}}{S_0^{(0)}} = \frac{\hat{S}_t^{(0)}}{\hat{S}_0^{(0)}} \tag{4.8}$$

for $t \in [0,T]$ and $S^{(0)}$ is the savings account process, see Karatzas & Shreve (1998). By (4.8) we can rewrite the fair pricing formula (4.7) as

$$\begin{aligned} U_{H_\tau}(t) &= E\left(\left(\frac{S_t^{(\delta_*)}}{S_\tau^{(\delta_*)}} \frac{S_\tau^{(0)}}{S_t^{(0)}}\right) \frac{S_t^{(0)}}{S_\tau^{(0)}} H_\tau \,\middle|\, \mathcal{A}_t\right) \\ &= E\left(\frac{\Lambda_\tau}{\Lambda_t} \frac{S_t^{(0)}}{S_\tau^{(0)}} H_\tau \,\middle|\, \mathcal{A}_t\right) \end{aligned}$$

for $t \in [0,T]$. If the Radon-Nikodym derivative process Λ is an $(\underline{\mathcal{A}}, P)$-martingale, then this relation leads by the Girsanov Theorem to the risk neutral pricing formula

$$U_{H_\tau}(t) = E_\theta\left(\frac{S_t^{(0)}}{S_\tau^{(0)}} H_\tau \,\middle|\, \mathcal{A}_t\right) \tag{4.9}$$

for $t \in [0,\tau]$. Here E_θ denotes expectation with respect to P_θ and the savings account $S^{(0)}$ is the numeraire. However, in a continuous benchmark model the Radon-Nikodym derivative process Λ may only be a strict $(\underline{\mathcal{A}}, P)$-local martingale. Therefore, we emphasize that such a model may *not* admit an equivalent risk neutral martingale measure P_θ. However, fair derivative prices can be always computed as conditional expectations directly under the real world probability measure p using the GOP $S^{(\delta_*)}$ as numeraire. This means, by relaxing the assumption on the existence of an equivalent risk neutral martingale measure, one can choose in a benchmark framework from a wider range of models than available under the standard risk neutral approach.

Additionally, we mention that in the case when a contingent claim H_T, which matures at a fixed date T, is independent of the GOP $S_T^{(\delta_*)}$, the fair pricing formula (4.7) yields the *actuarial pricing formula*

$$U_{H_T}(t) = P(t,T)\, E\left(H_T \,\middle|\, \mathcal{A}_t\right) \tag{4.10}$$

for $t \in [0,T]$. Here $P(t,T)$ denotes the corresponding fair zero coupon bond price

$$P(t,T) = E\left(\frac{S_t^{(\delta_*)}}{S_T^{(\delta_*)}} \,\middle|\, \mathcal{A}_t\right) \tag{4.11}$$

at time $t \in [0, T]$. Thus, the fair pricing concept generalizes not only standard risk neutral pricing but covers also the classical actuarial pricing. Note that the interest rate can be stochastic in (4.10) and (4.11), which is often not considered in the actuarial pricing literature.

5 The GOP as Best Performing Portfolio

5.1 Outperforming Growth Rate and Expected Return

The GOP can be considered to be the best performing portfolio in various ways. In the following, we describe some mathematical manifestations of this fact.

By relation (2.15) in Definition 2.2 it follows for any strictly positive portfolio process $S^{(\delta)}$ that at any time the growth rate $g_t^{\delta_*}$ of the GOP is never smaller than that of any other strictly positive portfolio. This yields a *first characterization*, which shows that the GOP outperforms the growth rate of all other portfolios.

From Corollary 4.1 we know that any strictly positive benchmarked portfolio $\hat{S}^{(\delta)}$ forms an $(\underline{\mathcal{A}}, P)$-supermartingale, which means that the expected return

$$E\left(\frac{\hat{S}^{(\delta)}_{t+h} - \hat{S}^{(\delta)}_t}{\hat{S}^{(\delta)}_t} \,\middle|\, \mathcal{A}_t\right) \leq 0 \tag{5.1}$$

of a benchmarked portfolio over any time period $[t, t+h] \subseteq [0, T]$ with $h > 0$ is always nonpositive. This yields a *second characterization* of outperformance, where the GOP, when used as benchmark, does not allow any nonnegative benchmarked portfolio to generate expected returns greater than zero.

5.2 Systematic Outperformance with Positive Probability

For an investor it is of interest to know whether or not it is possible to *systematically outperform* the GOP with some strictly positive probability by any other portfolio over any time period. To make this *third characterization* mathematically precise we introduce the following definition.

Definition 5.1 *A strictly positive portfolio $S^{(\delta)}$ is said to* systematically outperform with positive probability *another strictly positive portfolio $S^{(\bar{\delta})}$ if for some stopping times $\tau \in [0, T]$ and $\sigma \in [\tau, T]$ with*

$$S^{(\delta)}_\tau = S^{(\bar{\delta})}_\tau \tag{5.2}$$

and

$$S^{(\delta)}_\sigma \geq S^{(\bar{\delta})}_\sigma, \tag{5.3}$$

almost surely, it holds

$$P\left(S_\sigma^{(\delta)} > S_\sigma^{(\bar{\delta})} \,\middle|\, \mathcal{A}_\tau\right) > 0. \tag{5.4}$$

This means in the sense of Delbaen & Schachermayer (1995) that the GOP is a, so called, *maximal element*, which is important in the arbitrage pricing theory. According to the above definition, if a nonnegative portfolio systematically outperforms with positive probability the GOP, then it can generate over some time period wealth that is strictly greater than that accrued via the GOP with some strictly positive probability. We can now prove the following result.

Corollary 5.2 *No nonnegative portfolio systematically outperforms with positive probability the GOP.*

Proof: Consider a benchmarked, nonnegative portfolio $\hat{S}^{(\delta)} = \{\hat{S}_t^{(\delta)},\, t \in [0,T]\}$ with benchmarked value

$$\hat{S}_\tau^{(\delta)} = 1 \tag{5.5}$$

at a stopping time $\tau \in [0,T]$, almost surely, and assume for a later stopping time $\sigma \in [\tau, T]$ the inequality

$$\hat{S}_\sigma^{(\delta)} \geq 1, \tag{5.6}$$

almost surely. By the supermartingale property of $\hat{S}^{(\delta)}$, provided by Corollary 4.1, the optional sampling theorem and property (5.5) it follows that

$$0 \geq E\left(\hat{S}_\sigma^{(\delta)} - \hat{S}_\tau^{(\delta)} \,\middle|\, \mathcal{A}_\tau\right) = E\left(\hat{S}_\sigma^{(\delta)} - 1 \,\middle|\, \mathcal{A}_\tau\right) \geq 0. \tag{5.7}$$

Obviously, due to (5.7) and (5.6), the benchmarked value $\hat{S}_\sigma^{(\delta)}$ cannot be strictly greater than $\hat{S}_\tau^{(\delta)} = 1$ with any strictly positive probability. Thus, it follows by (5.6) almost surely that $\hat{S}_\sigma^{(\delta)} = 1$, which means that $S_\sigma^{(\delta)} = S_\sigma^{(\delta_*)}$. This proves by Definition 5.1 the Corollary 5.2. □

5.3 Outperforming the Long Term Growth Rate

Let us define for a strictly positive portfolio $S^{(\delta)}$ its *long term growth rate* $\tilde{g}_\delta$ as the almost sure limit

$$\tilde{g}_\delta \overset{\text{a.s.}}{=} \limsup_{T\to\infty} \frac{1}{T} \ln\left(\frac{S_T^{(\delta)}}{S_0^{(\delta)}}\right). \tag{5.8}$$

Note that this pathwise defined quantity does not involve any expectation. In the special case with constant short rate r and constant total market price for risk $|\theta|$ the GOP has by the law of large numbers the long term growth rate $\tilde{g}_{\delta_*} = r + \frac{|\theta|^2}{2}$.

The following result provides the outstanding property of the GOP that after sufficient long time its trajectory attains pathwise a value not less than that of any other strictly positive portfolio. This is a *fourth characterization* of outperformance, which expresses a most desirable feature of a portfolio for a long term investor. Since it can be argued that monetary authorities have a natural interest in a long term, pathwise outperformance of all strictly portfolios by the market portfolio of investable wealth, the following theorem supports the earlier discussed assumption that monetary authorities aim to maximize the growth of the total investable wealth in the market.

Theorem 5.3 *The GOP $S^{(\delta_*)}$ has almost surely the greatest long term growth rate compared with all other strictly positive portfolios $S^{(\delta)}$, that is*

$$\tilde{g}_{\delta_*} \geq \tilde{g}_{\delta}, \tag{5.9}$$

almost surely.

Proof: Similar to Karatzas & Shreve (1998) we consider a strictly positive portfolio $S^{(\delta)}$ with

$$S_0^{(\delta)} = S_0^{(\delta_*)}. \tag{5.10}$$

By Corollary 4.1 the strictly positive benchmarked portfolio $\hat{S}^{(\delta)}$ is an $(\underline{\mathcal{A}}, P)$-supermartingale. As a supermartingale it satisfies by (5.10) the inequality

$$\exp\{\varepsilon\, k\}\, P\left(\sup_{k \leq t < \infty} \hat{S}_t^{(\delta)} > \exp\{\varepsilon\, k\} \,\middle|\, \mathcal{A}_0\right) \leq E\left(\hat{S}_k^{(\delta)} \,\middle|\, \mathcal{A}_0\right) \leq \hat{S}_0^{(\delta)} = 1 \tag{5.11}$$

for all $k \in \{1, 2, \ldots\}$ and $\varepsilon \in (0, 1)$, see, for instance, Elliott (1982). For fixed $\varepsilon \in (0, 1)$ one finds

$$\sum_{k=1}^{\infty} P\left(\sup_{k \leq t < \infty} \ln\left(\hat{S}_t^{(\delta)}\right) > \varepsilon\, k \,\middle|\, \mathcal{A}_0\right) \leq \sum_{k=1}^{\infty} \exp\{-\varepsilon\, k\} < \infty. \tag{5.12}$$

Now, the Borel-Cantelli lemma implies the existence of a random variable K_ε such that

$$\ln\left(\hat{S}_t^{(\delta)}\right) \leq \varepsilon\, k \leq \varepsilon\, t$$

for all $k \geq K_\varepsilon$ and $t \geq k$, almost surely. Thus, one has almost surely

$$\sup_{T \geq k} \frac{1}{T} \ln\left(\hat{S}_T^{(\delta)}\right) \leq \varepsilon$$

for all $k \geq K_\varepsilon$, and therefore,

$$\limsup_{T \to \infty} \frac{1}{T} \ln\left(\frac{S_t^{(\delta)}}{S_0^{(\delta)}}\right) \leq \limsup_{T \to \infty} \frac{1}{T} \ln\left(\frac{S_T^{(\delta_*)}}{S_0^{(\delta_*)}}\right) + \varepsilon, \tag{5.13}$$

almost surely. Noting that relation (5.13) holds for all $\varepsilon \in (0, 1)$ it follows by (5.8) the inequality (5.9). □

Conclusion

The paper shows that the growth optimal portfolio (GOP) plays a central role in finance. It has been demonstrated via a portfolio selection theorem that investors, who maximize the drift of discounted portfolios with comparable levels of aggregate diffusion coefficients, form optimal portfolios which are linear combinations of the GOP and the savings account. The free parameter in the family of optimal portfolios is the risk aversion coefficient which can be exploited for modeling. The risk aversion coefficient controls the fraction of wealth invested in the savings account. It is illustrated that a standard expected utility maximization problem leads to an optimal portfolio strategy, which can be characterized via a corresponding risk aversion coefficient process. By assuming that investors prefer optimal portfolios, the market portfolio of investable wealth is shown to be an optimal portfolio and key results related to the capital asset pricing model, the Sharpe ratio and the Markowitz efficient frontier are easily derived. The GOP can be used in contingent claim pricing as natural numeraire together with the real world probability as pricing measure, generalizing risk neutral and actuarial pricing. Finally, it has been demonstrated in various ways how the GOP is the best performing portfolio.

A Appendix

Proof of the Portfolio Selection Theorem 3.3

Similarly as in Platen (2002) let us fix a time $t \in [0,T]$ and a constant $C > 0$. We optimize then the drift (3.4) $\alpha_\delta(t) = \sum_{k=1}^{d} \psi_\delta^k(t)\, \theta_t^k$ under the constraint (3.6), where

$$\sum_{k=1}^{d} (\psi_\delta^k(t))^2 = \gamma_t^\delta = C. \tag{A.1}$$

Using the Lagrange multiplier λ we maximize the function

$$G(\theta^1, \ldots, \theta^d, \psi_\delta^1, \ldots, \psi_\delta^d, C, \lambda) = \sum_{k=1}^{d} \psi_\delta^k\, \theta^k + \lambda \left(C - \sum_{k=1}^{d} \left(\psi_\delta^k\right)^2 \right). \tag{A.2}$$

For $\psi_\delta^1, \ldots, \psi_\delta^d$ to be optimal, it is necessary that the first order conditions

$$\frac{\partial G(\theta^1, \ldots, \theta^d, \psi_\delta^1, \ldots, \psi_\delta^d, C, \lambda)}{\partial \psi_\delta^k} = \theta^k - \lambda\, 2\, \psi_\delta^k = 0 \tag{A.3}$$

are satisfied for all $k \in \{1, 2, \ldots, d\}$. This means that for an optimal strategy we must have

$$\psi_\delta^k = \frac{\theta^k}{2\,\lambda} \tag{A.4}$$

for all $k \in \{1, 2, \ldots, d\}$. Using the constraint (A.1) we get from (A.4)

$$C = \sum_{k=1}^{d} \left(\psi_\delta^k\right)^2 = \left(\frac{|\theta|}{2\,\lambda}\right)^2. \tag{A.5}$$

We obtain by (A.4) and (A.5) the relation

$$\psi_\delta^k = \frac{\sqrt{C}}{|\theta|}\,\theta^k \tag{A.6}$$

for $k \in \{1, 2, \ldots, d\}$. From (3.3) and (A.6) it follows for $t \in [0, T]$ that

$$\pi_{\delta,t}^j = \frac{1}{\bar{S}_t^{(\delta)}} \sum_{k=1}^{d} \psi_\delta^k(t)\, b_t^{-1\,j,k} = \frac{\gamma_t^\delta}{\bar{S}_t^{(\delta)}\,|\theta_t|} \sum_{k=1}^{d} \theta_t^k\, b_t^{-1\,j,k} \tag{A.7}$$

for all $j \in \{1, 2, \ldots, d\}$. By (2.11) we then get

$$\pi_{\delta,t}^0 = 1 - \sum_{j=1}^{d} \pi_{\delta,t}^j = 1 - \frac{\gamma_t^\delta}{\bar{S}_t^{(\delta)}\,|\theta_t|} \left(1 - \pi_{\delta_*,t}^0\right) \tag{A.8}$$

for $t \in [0, T]$. Then it follows by (A.7) that

$$\gamma_t^\delta = \frac{1 - \pi_{\delta,t}^0}{1 - \pi_{\delta_*,t}^0}\, \bar{S}_t^{(\delta)}\,|\theta_t| \tag{A.9}$$

for all $t \in [0, T]$. With (3.3), (A.6) and (A.9) we then obtain (3.10). By (3.2), (A.6) and (A.9) we get (3.11). □

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