WIAS

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Iterative Construction of the Optimal Bermudan Stopping Time

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The Bermudan Pricing Problem

Consider a continuous time process L with state space $\mathbb{R}^D$. E.g. system of asset prices or Libor rates

Set of future dates $\mathbb{T} := \{\mathcal{T}_0, \mathcal{T}_1, \ldots, \mathcal{T}_k\}$ with $0 = \mathcal{T}_0 < \mathcal{T}_1 < \cdots < \mathcal{T}_k \leq T$.

Bermudan derivative: An option to exercise a cashflow $C_{\mathcal{T}_\tau} := C(\mathcal{T}_\tau, L(\mathcal{T}_\tau))$ at a future time $\mathcal{T}_\tau \in \mathbb{T}$, to be decided by the option holder

The $t = 0$ price in terms of a pricing numeraire B and EM Measure P,

$$V_0 = B(0) \sup_{\tau \in \{0,\ldots,k\}} E_P^{\mathcal{F}_0} \frac{C_{\mathcal{T}_\tau}}{B(\mathcal{T}_\tau)} =: B(0) \sup_{\tau \in \{0,\ldots,k\}} E^{\mathcal{F}_0} Z^{(\tau)}.$$

The supremum is over all $\mathbb{F}$-stopping indexes τ with $\mathbb{F} := \{\mathcal{F}_{\mathcal{T}_i}, 0 \leq i \leq k\}$, and $(\mathcal{F}_t, 0 \leq t \leq T)$ being the usual filtration generated by L.

Construction of the Optimal Bermudan Stopping Time

At a future time point t, when the option is not exercised before t, the Bermudan option value is given by

$$V_t = B(t) \sup_{\tau \in \{\kappa(t),\ldots,k\}} E^{\mathcal{F}_t} Z^{(\tau)}$$

with $\kappa(t) := \min\{m : \mathcal{T}_m \geq t\}$.
The process

$$Y_t^* := \frac{V_t}{B(t)},$$

called the *Snell-envelope* process, is a supermartingale. Indeed: Let $s < t$ and τ_t^* be an optimal stopping index at time t (which exists by general arguments), then it holds

$$E^{\mathcal{F}_s} Y_t^* = E^{\mathcal{F}_s} E^{\mathcal{F}_t} Z^{(\tau_t^*)} = E^{\mathcal{F}_s} Z^{(\tau_t^*)} \leq \sup_{\tau \in \{\kappa(s),\ldots,k\}} E^{\mathcal{F}_s} Z^{(\tau)} = Y_s^*$$

Canonical Solution by Backward Dynamic Programming

Denote $Y^{*(j)} := Y^*(\mathcal{T}_j)$, $\mathcal{F}^{(i)} := \mathcal{F}_{\mathcal{T}_i}$ and $\tau_i^* := \tau_{\mathcal{T}_i}^*$. Then:

At the last exercise date,

$$Y^{*(k)} = Z^{(k)}$$

and for $0 \leq j < k$,

$$Y^{*(j)} = \max\left(Z^{(j)}, E^{\mathcal{F}_j} Y^{*(j+1)}\right)$$

with

$$\tau_i^* = \inf\left\{j,\ i \leq j \leq k : Y^{*(j)} \leq Z^{(j)}\right\}.$$

$\longrightarrow$ Monte Carlo simulation would require N^k samples when conditional expectations are computed with N samples

Typically, N=10000, k=10 exercise opportunities, give 10^{40} samples !!

Improving a given family of stopping times

We consider a given stopping time family (τ_i), which satisfies

$$i \leq \tau_i \leq k, \; \tau_k = k,$$
$$\tau_i > i \Rightarrow \tau_i = \tau_{i+1}, \qquad 0 \leq i < k,$$

and the corresponding (lower bound) process Y,

$$Y^{(i)} := E^{\mathcal{F}^{(i)}} Z^{(\tau_i)} \leq Y^{*(i)}.$$

Examples:

▷ $\tau_i := \inf\{j \geq i : L(\mathcal{T}_j) \in G \subset \mathbb{R}^D\}$ Means: Exercise when the underlying process L enters a certain region G.

▷ The trivial family,

$$\tau_i \equiv i.$$ Means: Exercise immediately!

Other meaningful examples:

▷ The adapted stopping family

$$\tau_i := \inf\{l,\ i \le l \le k,\, E^{\mathcal{F}^{(i)}} Z^{(l)} = \max_{p:\, i \le p \le k} E^{\mathcal{F}^{(i)}} Z^{(p)}\},$$

which chooses at $\mathcal{T}_i$ the maximum of still-alive Europeans .
Exception: for this family $\tau_i > i \Rightarrow \tau_i = \tau_{i+1}$ is NOT true.

▷ The intuitively better strategy (stopping family)

$$\tau_i \;=\; \inf\{j : i \le j \le k,\, \max_{p:\, j \le p \le k} E^{\mathcal{F}^{(j)}} Z^{(p)} \le Z^{(j)}\},$$

which basically says: Wait until the cashflow is at least equal to the maximum of still-alive Europeans

One step improvement:

We introduce an intermediate process

$$\widetilde{Y}^{(i)} := \max_{p:\, i \le p \le k} E^{\mathcal{F}^{(i)}} Z^{(\tau_p)}.$$

Then, using $\widetilde{Y}^{(i)}$ as a new exercise criterion, we define

$$\begin{aligned} \widehat{\tau}_i : &= \inf\{j : i \le j \le k,\ \widetilde{Y}^{(j)} \le Z^{(j)}\} \\ &= \inf\{j : i \le j \le k,\ \max_{p:\, j \le p \le k} E^{\mathcal{F}^{(j)}} Z^{(\tau_p)} \le Z^{(j)}\}, \quad 0 \le i \le k, \end{aligned}$$

and consider the process

$$\widehat{Y}^{(i)} := E^{\mathcal{F}^{(i)}} Z^{(\widehat{\tau}_i)}$$

as a next approximation of the Snell envelope.

Key Proposition: It holds

$$Y^{(i)} \leq \widetilde{Y}^{(i)} \leq \widehat{Y}^{(i)} \leq Y^{*(i)}, \quad 0 \leq i \leq k.$$

Proof is based on induction from the last exercise date $\mathcal{T}_k$ down to $\mathcal{T}_0$, see paper

Iterative construction of the optimal stopping time

Start with some family of stopping times

$$\left(\tau_i^{(0)}\right)_{0\leq i\leq k},$$

satisfying

$$i \leq \tau_i^{(0)} \leq k, \quad \tau_k^{(0)} = k, \quad \tau_i \neq i \Rightarrow \tau_i = \tau_{i+1},$$

and the additional requirement,

$$Y^{0(i)} := E^{\mathcal{F}_i} Z^{(\tau_i^{(0)})} \geq Z^{(i)}, \quad 0 \leq i \leq k.$$

For example, start with $\tau_i^{(0)} \equiv i$.

Construction of the Optimal Bermudan Stopping Time

Suppose that for $m \geq 0$ the pair

$$\left((\tau_i^{(m)}), (Y^{m}(i)) \right)$$

is constructed with $\tau_i^{(m)}$ satisfying the previous requirements and

$$Y^{m}(i) := E^{\mathcal{F}_i} Z^{(\tau_i^{(m)})} \geq Z^{(i)}, \quad 0 \leq i \leq k.$$

Then define

$$\begin{aligned} \tau_i^{(m+1)} : &= \inf\{ j : i \leq j \leq k, \max_{p:\, j \leq p \leq k} E^{\mathcal{F}^{(j)}} Z^{(\tau_p^{(m)})} \leq Z^{(j)} \} \\ &=: \inf\{ j : i \leq j \leq k, \widetilde{Y}^{m+1}(j) \leq Z^{(j)} \}, \quad 0 \leq i \leq k, \end{aligned}$$

and set

$$Y^{m+1}(i) := E^{\mathcal{F}^{(i)}} Z^{(\tau_i^{(m+1)})}.$$

We thus have by the previous proposition

$$Z^{(i)} \le Y^{0(i)} \le Y^{m(i)} \le \widetilde{Y}^{m+1(i)} \le Y^{m+1(i)} \le Y^{*(i)}, \quad 0 \le m < \infty, \quad 0 \le i \le k.$$

and moreover for $m \ge 1$,

$$\tau_i^{(m)} \le \tau_i^{(m+1)} \le \tau_i^*,$$

for any optimal stopping time τ_i^*.

Next, we may take limits:

$$Y^{\infty(i)} := \text{(a.s.)} \lim_{m\uparrow\infty} \uparrow Y^{m(i)} \quad \text{and} \quad \tau_i^{\infty} := \text{(a.s.)} \lim_{m\uparrow\infty} \uparrow \tau_i^{(m)}, \quad 0 \le i \le k, \text{ and,}$$

$$Y^{\infty(i)} = \text{(a.s.)} \lim_{m\uparrow\infty} \uparrow E^{\mathcal{F}^{(i)}} Z^{(\tau_i^{(m)})} = E^{\mathcal{F}^{(i)}} Z^{(\tau_i^{\infty})}, \quad 0 \le i \le k.$$

by dominated convergence.

Main Theorem

The constructed limit process Y^{∞} coincides with the Snell envelope process Y^{*} and (τ_i^{∞}) acts as a family of optimal stopping times. We have

$$Y^{*(i)} = Y^{\infty(i)} = E^{\mathcal{F}^{(i)}} Z^{(\tau_i^{\infty})}, \quad 0 \le i \le k.$$

Proof: In the paper we prove that Y^{∞} is a supermartingale. The Th. then follows from

$$Z^{(i)} \le Y^{0(i)} \le Y^{\infty}, \text{ AND, } Y^{\infty} \le Y^{*},$$

Note: It even holds (see also Bender & Schoenmakers 2004)

$$Y^{m(i)} = Y^{*(i)} \quad \text{for} \quad m \ge k - i$$

$\longrightarrow$ After k iterations the Snell Envelope is attained!

Upper approximations of the Snell envelope by Duality

The Dual Method

Consider again the discrete filtration $\left(\mathcal{F}^{(j)}\right)_{j=0,\dots,k}$ with $\mathcal{F}^{(j)} := \mathcal{F}_{\mathcal{T}_j}$, $0 \le j \le k$, $\mathcal{F}^{(0)} := \mathcal{F}_0$, and a discrete martingale $(M_j)_{j=0,\dots,k}$ with $M_0 = 0$ with respect to this filtration. Following Rogers (2001) we observe that

$$\begin{aligned} Y_0 &= \sup_{\tau \in \{0,\dots,k\}} E^{\mathcal{F}_0} \frac{C_{\mathcal{T}_\tau}}{B(\mathcal{T}_\tau)} =: \sup_{\tau \in \{0,\dots,k\}} E^{\mathcal{F}_0} Z^{(\tau)} = \sup_{\tau \in \{0,,\dots,k\}} E^{\mathcal{F}_0} \left[Z^{(\tau)} - M_\tau \right] \\ &\le E^{\mathcal{F}_0} \max_{0 \le j \le k} \left[Z^{(j)} - M_j \right] \end{aligned}$$

Hence the r.h.s. gives an upper bound for the Bermudan price $V_0 = B(0) Y_0$.

Theorem (Davis Karatzas (1994), Rogers (2001), Haugh & Kogan (2001))

Let us consider the Snell envelope process Y at the discrete time set $\{\mathcal{T}_0, ..., \mathcal{T}_k\}$, and define $Y^{(j)} := Y(\mathcal{T}_j)$, $0 \leq j \leq k$, $Y^{(0)} := Y_0$. Let further M^Y be the (unique) Doob-Meyer martingale part of $\left(Y^{(j)}\right)_{0 \leq j \leq k}$, i.e. M^Y is an $\left(\mathcal{F}^{(j)}\right)$-martingale which satisfies

$$Y^{(j)} = Y_0 + M_j^Y - F_j^Y, \quad j = 0, ..., k,$$

with $M_0^Y := F_0^Y := 0$ and F^Y being such that F_j^Y is $\mathcal{F}^{(j-1)}$ measurable for $j = 1, ..., k$. Then we have

$$Y_0 = E^{\mathcal{F}_0} \max_{0 \leq j \leq k} \left[Z^{(j)} - M_j^Y \right].$$

Proof

Note that always $Y_j \geq Z^{(j)}$ and that F_j^Y is nondecreasing since $(Y^{(j)})$ is an $\left(\mathcal{F}^{(j)}\right)$-supermartingale. Hence

$$\begin{aligned} Y_0 \;&\leq\; E^{\mathcal{F}_0} \sup_{0\leq j\leq k}\left[Z^{(j)} - M_j^Y\right] = E^{\mathcal{F}_0}\left\{Y_0 + \sup_{0\leq j\leq k}\left[Z^{(j)} - Y^{(j)} - F_j^Y\right]\right\} \\ &\leq\; E^{\mathcal{F}_0}\left\{Y_0 + \sup_{0\leq j\leq k}\left[-F_j^Y\right]\right\} = Y_0 - F_0^Y = Y_0 \qquad \square \end{aligned}$$

Note: There is a recent Dual method by Jamshidian due to Multiplicative Doob Meyer decompostion of Y.

From lower to upper bound by Duality

Consider a lower approximation $\overline{Y}$, hence $\overline{Y} \leq Y^*$.

Let $\overline{M}$ be the martingale part of the Doob-Meyer decomposition of $\overline{Y}$, hence

$$\overline{M}^{(0)} = 0;$$

$$\overline{M}^{(j)} = \overline{M}^{(j-1)} + \overline{Y}^{(j)} - E^{\mathcal{F}^{(j-1)}}\overline{Y}^{(j)} = \sum_{l=i+1}^{j} \overline{Y}^{(l)} - \sum_{l=i+1}^{j} E^{\mathcal{F}^{(l-1)}}\overline{Y}^{(l)}, \; j = 1, \ldots, k.$$

Then, by Duality

$$Y^{*(0)} \leq E^{\mathcal{F}_0} \max_{0 \leq j \leq k} \left(Z^{(j)} - \overline{M}^{(j)} \right) =: \overline{Y}_{up}^{(0)}.$$

The gap between $\overline{Y}^{(0)}$ and $\overline{Y}_{up}^{(0)}$ depends, in some sense, on how far the lower bound process $\overline{Y}$ is away from being a supermartingale:

Theorem (Kol., Sch. 2003)

Suppose, that $Y^{*(0)} \geq \overline{Y}^{(0)}$ and that $\overline{Y}^{(i)} \geq Z^{(i)}$, $i = 0, \ldots, k$. Then,

$$0 \leq \overline{Y}_{up}^{(0)} - \overline{Y}^{(0)} \leq E^{\mathcal{F}_0} \sum_{j=0}^{k-1} \max(E^{\mathcal{F}_j} \overline{Y}^{(j+1)} - \overline{Y}^{(j)},\ 0).$$

Proof:

$$
\begin{aligned}
\overline{Y}^{(0)}_{up} &= E^{\mathcal{F}_0} \max_{0\le j\le k} \Big(Z^{(j)} - \sum_{l=1}^{j} \overline{Y}^{(l)} + \sum_{l=1}^{j} E^{\mathcal{F}^{(l-1)}} \overline{Y}^{(l)}\Big) \\
&= \overline{Y}^{(0)} + E^{\mathcal{F}_0} \max_{0\le j\le k} \Big(Z^{(j)} - \overline{Y}^{(j)} + \sum_{l=0}^{j-1} E^{\mathcal{F}^{(l)}} (\overline{Y}^{(l+1)} - \overline{Y}^{(l)})\Big) =: \overline{Y}^{(0)} + \Delta^{(0)}.
\end{aligned}
$$

So,

$$
\begin{aligned}
\Delta^{(0)} = \overline{Y}^{(0)}_{up} - \overline{Y}^{(0)} &\le E^{\mathcal{F}_0} \max_{0\le j\le k} \sum_{l=0}^{j-1} \big(E^{\mathcal{F}^{(l)}} \overline{Y}^{(l+1)} - \overline{Y}^{(l)}\big) \\
&\le E^{\mathcal{F}_0} \max_{0\le j\le k} \sum_{l=0}^{j-1} \max\big(E^{\mathcal{F}^{(l)}} \overline{Y}^{(l+1)} - \overline{Y}^{(l)}, 0\big) \\
&\le E^{\mathcal{F}_0} \sum_{j=0}^{k-1} \max\big(E^{\mathcal{F}^{(j)}} \overline{Y}^{(j+1)} - \overline{Y}^{(j)}, 0\big). \qquad \square
\end{aligned}
$$

Construction of the Optimal Bermudan Stopping Time

Consider our iterative sequence of lower bound processes Y^m, $m = 0, 1, 2, \ldots$,

$$Y^{m(i)} \uparrow Y^{*(i)}$$

We now deduce a sequence of upper bound processes:

$$Y_{up}^{m(i)} \;:=\; E^{\mathcal{F}_i} \max_{i \le j \le k} \Big(Z^{(j)} - \sum_{l=i+1}^{j} Y^{m(l)} + \sum_{l=i+1}^{j} E^{\mathcal{F}^{(l-1)}} Y^{m(l)} \Big) =: Y^{m(i)} + \Delta^{m(i)}.$$

By the previous theorem,

$$0 \le \Delta^{m(i)} \le E^{\mathcal{F}_i} \sum_{j=i}^{k-1} \max \left(E^{\mathcal{F}_j} Y^{m(j+1)} - Y^{m(j)},\; 0 \right).$$

By letting $m \uparrow \infty$ on the r.h.s., (a.s.) $\lim_{m \to \infty} \Delta^{m(i)} = 0, \quad 0 \le i \le k$. Hence, the sequence Y_{up}^m converges to the Snell envelope also, i.e.,

$$\text{(a.s.)} \lim_{m \to \infty} Y_{up}^{m(i)} = \text{(a.s.)} \lim_{m \to \infty} Y^{m(i)} = Y^{*(i)}, \quad 0 \le i \le k.$$

A numerical example: Bermudan swaptions in the LIBOR market model

Consider the Libor Market Model with respect to a tenor structure $0 < T_1 < T_2 < \ldots < T_n$, e.g. in the spot Libor measure P^* induced by the numeraire

$$B^*(t) := \frac{B_{m(t)}(t)}{B_1(0)} \prod_{i=0}^{m(t)-1} (1 + \delta_i L_i(T_i))$$

with $m(t) := \min\{m : T_m \geq t\}$.
The dynamics of the forward Libor $L_i(t)$ is given by a system of SDE's

$$dL_i = \sum_{j=m(t)}^{i} \frac{\delta_j L_i L_j \, \gamma_i \cdot \gamma_j}{1 + \delta_j L_j} \, dt + L_i \, \gamma_i \cdot dW^*.$$

Here $\delta_i = T_{i+1} - T_i$ are day count fractions, and

$$t \to \gamma_i(t) = (\gamma_{i,1}(t), \ldots, \gamma_{i,d}(t))$$

are deterministic volatility vector functions defined in $[0, T_i]$, called factor loadings.

A (payer) Swaption over a period $[T_i, T_n]$, $1 \leq i \leq k$. A *swaption contract* with maturity T_i and strike θ with principal $\$1$ gives the right to contract at T_i for paying a fixed coupon θ and receiving floating Libor at the settlement dates T_{i+1},...,T_n. So by this definition, its cashflow at maturity is

$$S_{i,n}(T_i) := \left(\sum_{j=i}^{n-1} B_{j+1}(T_i)\delta_j \left(L_j(T_i) - \theta \right) \right)^+ .$$

A Bermudan Swaption gives the the right to exercise a cashflow

$$C_{T_\tau} := S_{\tau,n}(T_\tau)$$

at an exercise date $T_\tau \in \{T_1, \ldots, T_n\}$ to be decided by the option holder.

10 yr. Bermudan swaption: Comparison of lower bound iterations

$$Y^1, \quad Y^2, \quad Y^{1,up}, \quad Y_{\text{Andersen}}, \quad Y^{up}_{\text{Andersen}}$$

θ	d	$Y_0^{(1)}$ (SD)	$Y_0^{(2)}$ (SD)	$Y_0^{(1),up}$ (SD)	$Y_{A,0}$ (SD)	$Y_{A,0}^{up}$ (SD)
0.08 (ITM)	1	1104.6(0.5)	1108.9(2.4)	1109.4(0.7)	1107.7(0.5)	1109.0(0.5)
	2	1098.6(0.4)	1100.5(2.4)	1103.7(0.7)	1097.5(0.4)	1104.1(0.6)
	10	1094.4(0.4)	1096.9(2.1)	1098.1(0.6)	1093.0(0.4)	1099.5(0.6)
	40	1093.6(0.4)	1096.1(2.0)	1096.6(0.6)	1092.9(0.4)	1098.2(0.5)
0.10 (ATM)	1	374.3(0.4)	381.2(1.6)	382.9(0.8)	381.2(0.4)	383.1(0.4)
	2	357.9(0.3)	364.4(1.5)	366.4(0.8)	354.6(0.4)	367.4(0.6)
	10	337.8(0.3)	343.5(1.3)	345.6(0.7)	331.9(0.3)	348.2(0.6)
	40	332.6(0.3)	338.7(1.2)	341.2(0.8)	327.0(0.3)	342.7(0.6)
0.12 (OTM)	1	119.0(0.2)	121.0(0.6)	121.3(0.4)	120.5(0.2)	121.1(0.1)
	2	112.7(0.2)	113.8(0.5)	114.9(0.4)	110.0(0.2)	114.4(0.3)
	10	100.2(0.2)	100.7(0.4)	101.5(0.3)	95.7(0.2)	102.1(0.3)
	40	96.5(0.2)	96.9(0.4)	97.7(0.3)	92.2(0.2)	98.1(0.3)

Computation times in minutes: (3 months Euler stepping of the LIBOR SDE, Pentium IIII)

θ	d	$Y_0^{(1)}$	$Y_0^{(2)} - Y_0^{(1)}$	$Y_0^{(1),up} - Y_0^{(1)}$	$Y_{A,0}$	$Y_{A,0}^{up} - Y_{A,0}$
0.0 (ITM)	0.2	1.5	2.1	0.6	2.5	3.0
	0.4	1.5	2.2	0.6	2.3	2.9
	2.0	1.5	2.6	0.7	2.5	3.5
	8.0	2.1	5.4	6.1	3.6	6.0
0.0 (ATM)	0.2	4.3	5.4	1.0	5.3	5.2
	0.4	4.3	7.7	1.1	5.3	5.3
	2.0	4.9	9.5	5.1	6.1	6.6
	8.0	6.9	15.9	2.3	9.1	11.5
0.0 (OTM)	0.2	6.1	2.0	1.4	6.8	6.7
	0.4	6.3	1.7	1.4	6.8	6.8
	2.0	7.3	2.0	2.3	7.9	8.7
	8.0	12.7	3.5	4.3	11.7	15.6

Conclusions from the tables:

▷ For more than 1 factor the computed lower bound $Y^{2(0)}$, hence the second iteration, is usually found above the middle of $Y_A^{(0)}$ (Andersen's lower bound) and its Dual upper bound $Y_{A,up}^{(0)}$.

▷ Computation times may be considered low in view of the high-dimensionality of the problem!

More general Conclusions

The iterative approach provides a general method for improving any given stopping time τ with some natural properties.

E.g: One can start with Andersen's process Y_A and iterate from there.

Generic Method for Discrete Optimal Stopping

- The implementation of the proposed iterative procedure is straightforward, and can be done in a generic way for a variety of (not necessarily financial) optimal stopping problems.
- Although the algorithm gives rise to nested Monte Carlo simulations, by a proper implementation the second iteration can be computed suprisingly fast;

couple of minutes for Bermudans in a full blown Libor model!

Outlook

(A) The iteration procedure is extended to

multiple callable structures $\longrightarrow$ Bender & Schoenmakers 2004

(B) We expect that, when the producers of microprocessor chips keep

"Riding the Exponential",

computation of higher order iterations by this generic method, practically exact prices, will become feasible in the near future.